MATH3360: Mathematical Imaging

Chapter 3: Image Enhancement in the Frequency Domain

Image enhancement refers to the process, by which we improve the image so that it looks subjectively better. We do not really know what the image should look like, but we can tell whether it has been improved or not, by considering, for example, whether more detail can be seen, or whether unwanted flickering has been removed, or the contrast is better.

Image enhancement is broadly defined. It can be referred to as different imaging tasks, which include image denoising, image deblurring, image sharpening and so on. In this chapter, some image enhancement tasks will be discussed.

Image denoising aims to restore an image corrupted by noise. Below is a simple example.

The image on the left is a noisy image. The image on the right is the restored image. The noises are removed using a mathematical model (which we are going to learn in this chapter).

Image deblurring aims to restore an image, whose features are blurry due to various factors (e.g. motion, atmospheric turbulence). Here is a simple example of motion blur.

In the above figure, the left shows a blurry image caused by motion (e.g. taking the photo on a moving car). The right shows the restored image using a mathematical model (which we are going to learn in this chapter).

Image enhancement can either be done in the spatial domain or the frequency domain. In this chapter, we will firstly look at how image enhancement can be done in the frequency domain.

Simply speaking, image processing in the frequency domain is done as follows. The image is firstly transformed using discrete Fourier transform (which can be computed efficiently using fast Fourier transform (FFT)). The Fourier coefficients are then adjusted according to different imaging tasks. The restored image can be obtained by taking the inverse Fourier transform (which can again be computed efficiently using FFT). In fact, instead of using DFT, wavelet transform (such as Haar transform) can also be applied.

1 Image denoising in the frequency domain

We will first discuss image denoising algorithms in the frequency domain.

1.1 Image denoising by low-pass filtering

Our goal in this subsection is as follows.

Goal:

1. Remove high frequency component (low pass filter) for image de-noising.
2. Remove low frequency component (high pass filter) for extraction of image details.

In order to denoise an image in the frequency domain, we need to identify the high/low frequency components of the DFT of f.

Let F be a $N \times N$ image, where N is even. Let $\hat{F}$ be the DFT of F.

High / Low frequency components of $\hat{F}$

Recall:

$$\hat{F}(k,l) = \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} F(m,n) e^{-j\frac{2\pi}{N}(mk+nl)}$$

$\hat{F}(k,l)$ is called the Fourier coefficient at (k,l).

Observe that: for $0 \leq k,l \leq \dfrac{N}{2} - 1$

$$\begin{aligned}
\hat{F}\left(\frac{N}{2}+k, \frac{N}{2}+l\right) &= \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} F(m,n) e^{-j\frac{2\pi}{N}(m(\frac{N}{2}+k)+n(\frac{N}{2}+l))} \\
&= \frac{1}{N^2} \overline{\sum_{m=0}^{N-1} \sum_{n=0}^{N-1} F(m,n)(-1)^{m+n} e^{-j\frac{2\pi}{N}(m(-k)+n(-l))}} \\
&= \frac{1}{N^2} \overline{\sum_{m=0}^{N-1} \sum_{n=0}^{N-1} F(m,n) e^{-j\frac{2\pi}{N}(m(\frac{N}{2}-k)+n(\frac{N}{2}-l))}} \\
&= \overline{\hat{F}\left(\frac{N}{2}-k, \frac{N}{2}-l\right)}
\end{aligned}$$

Remark. *Computing part of $\hat{F}$ can determine the rest!*

From above, we deduce that:

$$\begin{aligned}F(m,n) =& \sum_{0\le k,l\le \frac{N}{2}-1}\left[\hat{F}\left(\frac{N}{2}+k,\frac{N}{2}+l\right)e^{j\frac{2\pi}{N}[(\frac{N}{2}+k)m+(\frac{N}{2}+l)n]}\right]\\
&+\sum_{1\le k,l\le \frac{N}{2}-1}\left[\overline{\hat{F}\left(\frac{N}{2}+k,\frac{N}{2}+l\right)}e^{j\frac{2\pi}{N}[(\frac{N}{2}-k)m+(\frac{N}{2}-l)n]}\right]\\
&+\sum_{0\le k,l\le \frac{N}{2}-1}\left[\hat{F}\left(\frac{N}{2}+k,\frac{N}{2}-l\right)e^{j\frac{2\pi}{N}[(\frac{N}{2}+k)m+(\frac{N}{2}-l)n]}\right]\\
&+\sum_{1\le k,l\le \frac{N}{2}-1}\left[\overline{\hat{F}\left(\frac{N}{2}+k,\frac{N}{2}-l\right)}e^{j\frac{2\pi}{N}[(\frac{N}{2}-k)m+(\frac{N}{2}+l)n]}\right]\\
&+\sum_{0\le l\le \frac{N}{2}-1}\hat{F}\left(0,\frac{N}{2}+l\right)e^{j\frac{2\pi}{N}[(\frac{N}{2}+l)n]}+\sum_{1\le l\le \frac{N}{2}-1}\overline{\hat{F}\left(0,\frac{N}{2}+l\right)}e^{j\frac{2\pi}{N}[(\frac{N}{2}-l)n]}\\
&+\sum_{0\le k\le \frac{N}{2}-1}\hat{F}\left(\frac{N}{2}+k,0\right)e^{j\frac{2\pi}{N}[(\frac{N}{2}+k)m]}+\sum_{1\le k\le \frac{N}{2}-1}\overline{\hat{F}\left(\frac{N}{2}+k,0\right)}e^{j\frac{2\pi}{N}[(\frac{N}{2}-k)m]}+\hat{F}(0,0)\end{aligned}$$

Observations:

- When k and l are close to $\frac{N}{2}$, $\hat{F}\left(\underbrace{\frac{N}{2}+k}_{\approx N},\underbrace{\frac{N}{2}+l}_{\approx N}\right)$ is associated to $\underbrace{e^{-j\frac{2\pi}{N}(k'm+l'n)}}_{\approx e^{-j\frac{2\pi}{N}(\tilde{k}m+\tilde{l}n)}}$, where k' and l' are close to N and $\tilde{k}$ and $\tilde{l}$ are close to 0.

  Therefore, Fourier coefficients at bottom right corner are associated to low frequency components.
- Similarly, we can check Fourier coefficients at the four corners are associated to low frequency components.
- On the other hand, Fourier coefficients in the middle are associated to high frequency components.

Remark.
- *High pass filtering = Remove Fourier coefficients at the four corners*
- *Low pass filtering = Remove Fourier coefficients in the middle.*

Centralization of Frequency domain

Let $F(x,y)$ be a $N\times N$ image with $0\le x\le N-1, 0\le y\le N-1$
Let $\hat{F}(u,v)$ be the DFT of F with $0\le u\le N-1, 0\le v\le N-1$

Note that:

- High-frequency components are located near $\left(\frac{N}{2},\frac{N}{2}\right)$.
- Low-frequency components are located near 4 corners.

So, if we let $\tilde{F}(u,v)=\hat{F}\left(u-\frac{N}{2},v-\frac{N}{2}\right)$ where $0\le u\le N-1, 0\le v\le N-1$, then

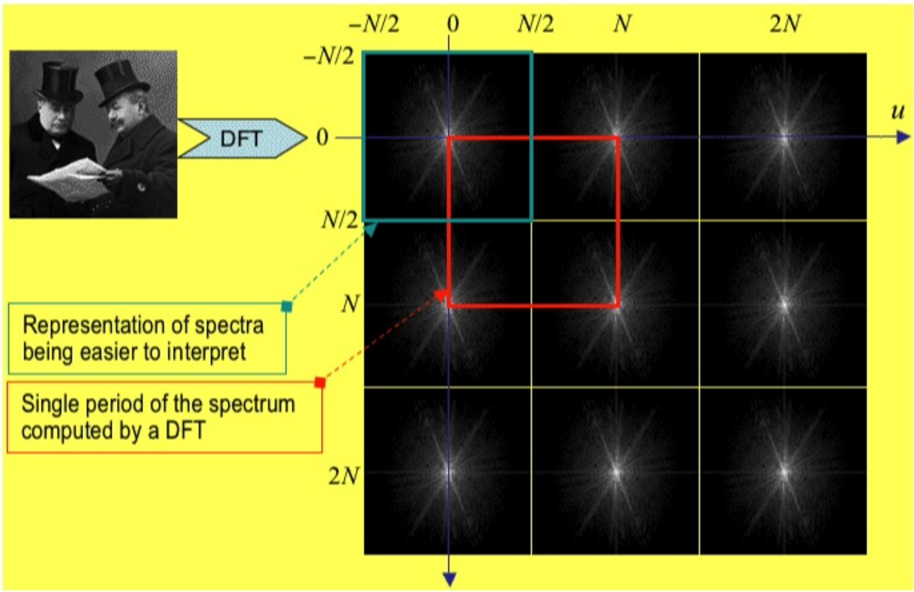
- High-frequency components are located at the four corners of $\tilde{F}(u,v)$.
- Low-frequency components are located at the middle part of $\tilde{F}(u,v)$.

Consider the discrete Fourier transform of $(-1)^{x+y}F(x,y)$:

$$\begin{aligned}
&DFT(F(x,y)(-1)^{x+y})(u,v)\\
=&\frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)e^{j\pi(x+y)}exp\left(-j2\pi\left(\frac{ux}{N}+\frac{vy}{N}\right)\right)\\
=&\frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)exp\left(-j2\pi\left(\frac{(u-N/2)x}{M}+\frac{(v-N/2)y}{N}\right)\right)\\
=&\hat{F}\left(u-\frac{N}{2},v-\frac{N}{2}\right)
\end{aligned}$$

Therefore, to compute $\tilde{F}(u,v)$, we can compute DFT of $(-1)^{x+y}f(x,y)$.

The idea of centralization is illustrated below:



After centralization, the blue window is considered.

Definition 1.1. A **low-pass filter (LPF)** leaves low frequencies unchanged, while attenuating the high frequencies.

A **high-pass filter (HPF)** leaves high frequencies unchanged, while attenuating the low frequencies.

Basic steps of filtering in the frequency domain

1. Multiply $f(x,y)$ by $(-1)^{x+y}$, i.e. $\tilde{f}(x,y)=(-1)^{x+y}f(x,y)$.
2. Compute $\tilde{F}(u,v)=DFT(\tilde{f})(u,v)$.
3. Multiply $\tilde{F}$ by a real "filter" function $\tilde{H}(u,v)$ to get
$$G(u,v)=\tilde{H}(u,v)\tilde{F}(u,v)$$
(point-wise multiplication, but not matrix multiplication)
4. Compute inverse DFT of $G(u,v)$.

5. Take real part of the result in Step 4.
6. Multiply the result in Step 5 by $(-1)^{x+y}$.

Remark. *1. H is taken to either remove low-frequency or high-frequency components.*

2. In the spatial domain,

$$\mathcal{F}^{-1}(G) = g = N^2\mathcal{F}^{-1}(H) * \mathcal{F}^{-1}(\tilde{F}) = N^2 h * \tilde{f}$$

Hence, filtering in frequency domain $\Leftrightarrow$ Linear filtering in spatial domain.

Examples of Low-Pass filters for image denoising

Note: From now on, we will assume we work on the centered spectum. That is, we consider $\hat{F}(u,v)$ where $-\frac{N}{2} \leq u \leq \frac{N}{2} - 1$ and $-\frac{N}{2} \leq v \leq \frac{N}{2} - 1$.

1. Ideal low-pass filter (ILPF):

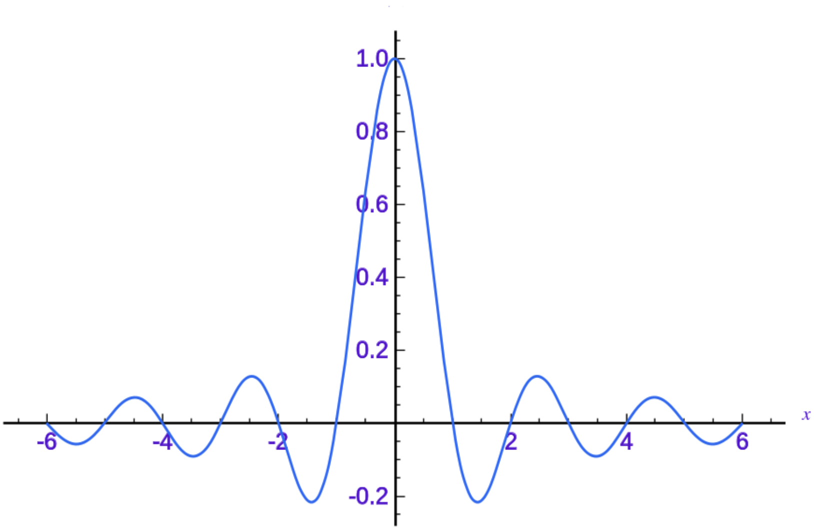
$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) := dist(u, M\mathbb{Z})^2 + dist(v, N\mathbb{Z})^2 \leq D_0^2 \\ 0 & \text{if } D(u,v) > D_0^2 \end{cases},$$

where for any $x \in \mathbb{Z}$ and $y \in \mathbb{N} \setminus \{0\}$,

$$dist(x, y\mathbb{Z}) = \min\{|x - ky| : k \in \mathbb{Z}\} = \begin{cases} x - \lfloor \frac{x}{y} \rfloor y = \text{mod}_y(x) & \text{if } \text{mod}_y(x) \leq \frac{y}{2}, \\ \lceil \frac{x}{y} \rceil y - x = y - \text{mod}_y(x) & \text{if } \text{mod}_y(x) \geq \frac{y}{2}. \end{cases},$$

where the value of $\text{mod}_y(x)$ is taken from $\{0, 1, \cdots, y-1\}$ (agrees with the topological definition of distance between a point and a set).

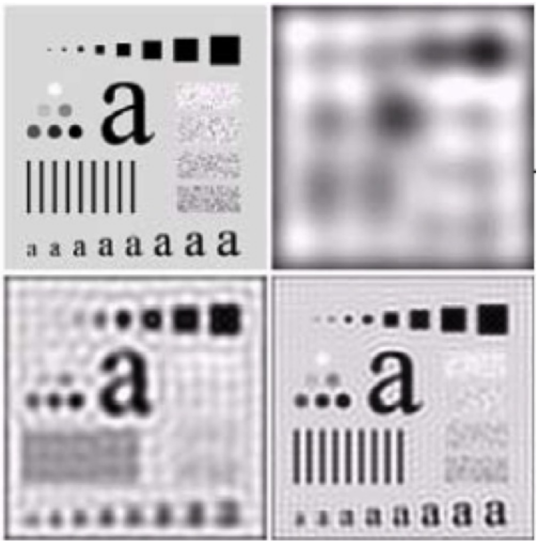
In the spatial domain, the filter looks like (in one dimensional case):



Good: Simple

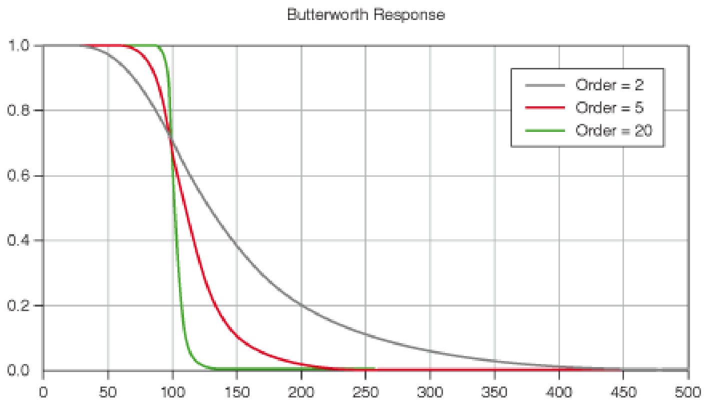
Bad: Produce the ringing effect (A pixel can be affected by pixels far away from it)

Ideal low-pass filter is applied to the following image with different D_0. Ringing effect is obviously observed.



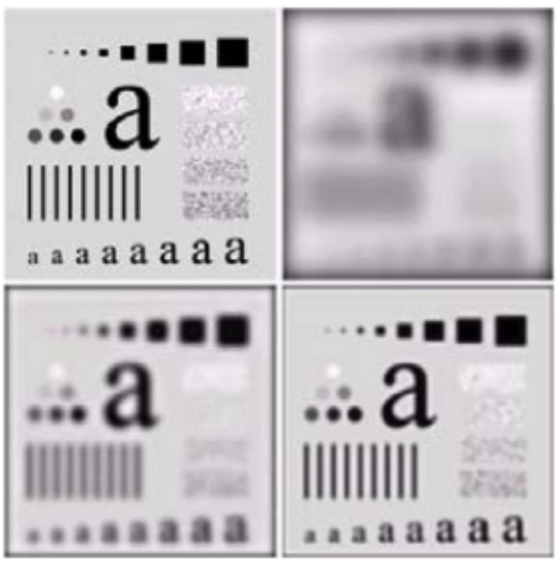
2. Butterworth low-pass filter (BLPF) of order n ($n \geq 1$ integer):

$$H(u,v) = \frac{1}{1 + (D(u,v)/D_0^2)^n}$$



Good: Produce less (or no visible) ringing effect if the order n is carefully chosen.

Butterworth low-pass filter is applied to the following image with different D_0 and n. No visible ringing is observed.



3. Gaussian low-pass filter:

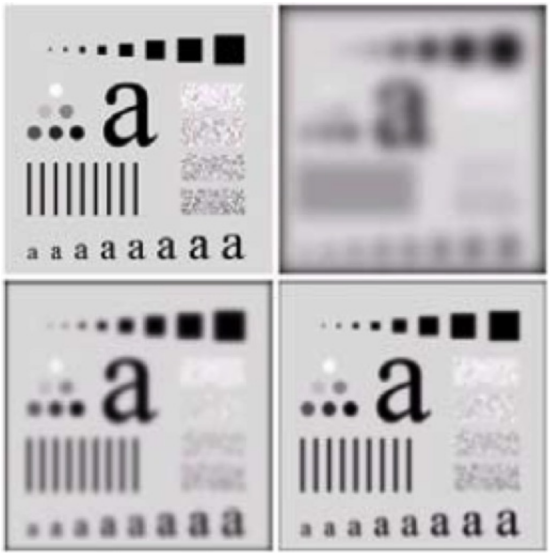
$$H(u,v) = exp\left(-\frac{D(u,v)}{2\sigma^2}\right)$$

σ is called the spread of the Gaussian function.

Good: Produce no visible ringing.

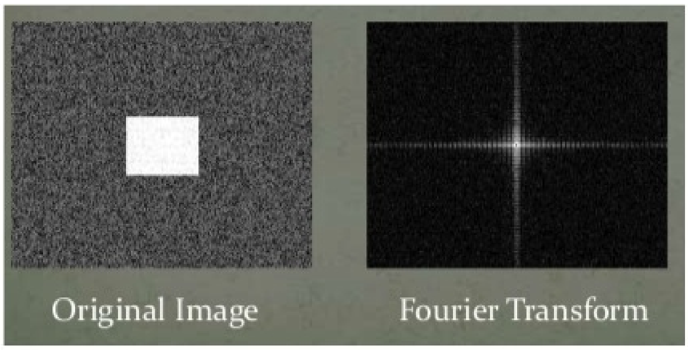
Why? Inverse DFT of a Gaussian function is also Gaussian.
Therefore, no visible ringing effect.

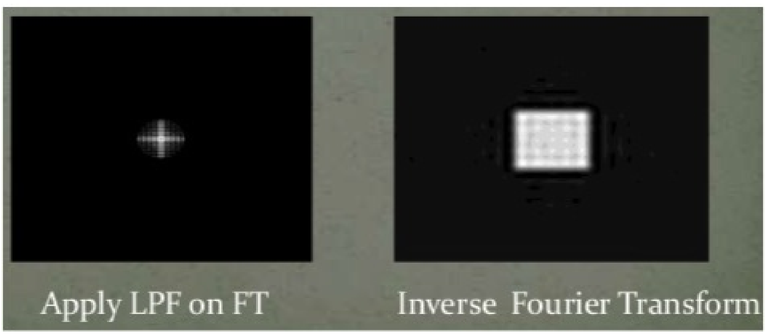
Gaussian low-pass filter is applied to the following image with different D_0 and n. No visible ringing is observed.



Experimental results of image denoising by low pass filtering in the frequency domain

Image denoising using ideal low-pass filter in the frequency domain:





Note that ringing is obviously observed.

Image denoising using butterworth low-pass filter in the frequency domain:

Image denoising using Gaussian low-pass filter in the frequency domain:

Examples of high-pass filter:

1. Ideal high-pass filter (IHPF):

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \leq D_0^2 \\ 1 & \text{if } D(u,v) > D_0^2 \end{cases}$$

   Bad: Produce ringing effect

2. Butterworth high-pass filter (BHPF) of order n:

$$H(u,v) = \frac{1}{1 + (D_0^2/D(u,v))^n}$$

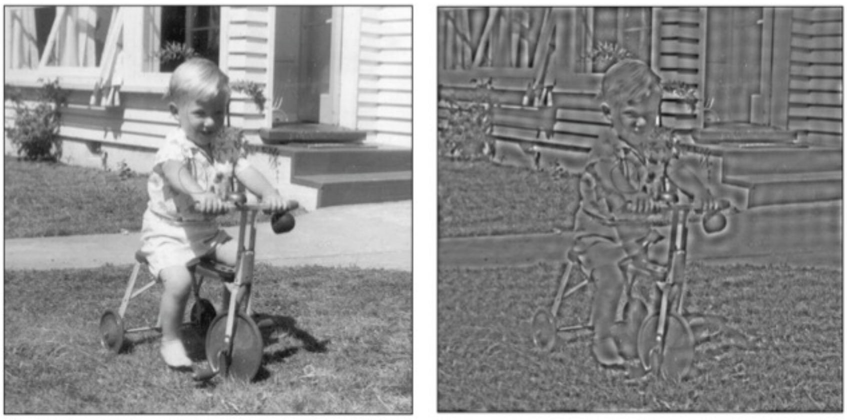
   Good: Less ringing.

3. Gaussian high-pass filter:

$$H(u,v) = 1 - exp\left(-\frac{D(u,v)}{2\sigma^2}\right)$$

   Good: No ringing.

The ideal high-pass filter is applied on the following image on the left. The result after the high-pass filter is shown on the right:

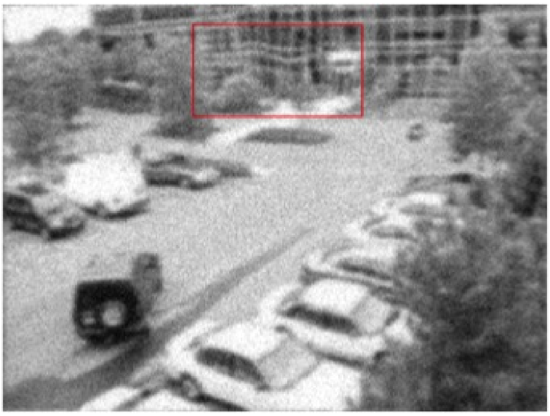


2 Image deblurring in the frequency domain

2.1 Basic idea of image deblurring

Observation: Image can be degraded due to motion, turbulence, out of focus and so on. Below are some examples of blurred/degraded images.

Below shows an example of degraded image by the atmospheric turbulence:



Below shows an example of motion blur:

Goal: We would like to model image blur/degradation

In general, an observed image g can be modeled as:

$$g = H(f) + n$$

where H is the degradation function/operator and n is the additive noise.

Assumption on H:

1. H is position invariant: Let $g(x,y) = H(f)(x,y)$ and let $\tilde{f}(x,y) := f(x-\alpha, y-\beta)$. Then:

$$H(\tilde{f})(x,y) = g(x-\alpha, y-\beta)$$

2. Linear: $H(f_1+f_2) = H(f_1)+H(f_2)$ and $H(\alpha f) = \alpha H(f)$ where α is a scalar multiplication.

With the above assumptions, image degradation is in fact a convolution. Consider an impluse signal:

$$\delta(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,0) \\ 0 & \text{if } (x,y) \neq (0,0) \end{cases}$$

Then,

$$f(x,y) = f * \delta(x,y) = \sum_{\alpha=-\frac{M}{2}}^{\frac{M}{2}-1} \sum_{\beta=-\frac{N}{2}}^{\frac{N}{2}-1} f(\alpha,\beta)\delta(x-\alpha, y-\beta)$$

$$\begin{aligned} \therefore g(x,y) &= H(f)(x,y) \\ &= \sum_{\alpha}\sum_{\beta} f(\alpha,\beta)H(\delta)(x-\alpha,y-\beta) \quad \text{by linearity and position invariant} \\ &= \sum_{\alpha}\sum_{\beta} f(\alpha,\beta)h(x-\alpha,y-\beta) \quad \text{where } h(x,y) = H(\delta)(x,y) \\ &= f * h(x,y) \end{aligned}$$

Hence, the degradation with the above assumption is actually a convolution.

Remark.

1. *Recall that h is called the point spread function.*
2. *In general, an observed image can be modelled as:*

$$g(x,y) = h * g(x,y) + n(x,y)$$

In the frequency domain, we have:

$$G(u,v) = cH(u,v)F(u,v) + N(u,v)$$

for some constant c.

2.2 Examples of degradation functions

1. Atmospheric turbulence blur:
   In frequency domain, define:

$$H(u,v) = exp\left(-k(u^2+v^2)^{5/6}\right)$$

   where k = degree of turbulence.

   Remark. *Usually,*
   $k = 0.0025$*: severe turbulence*
   $k = 0.001$*: mild turbulence*
   $k = 0.00025$*: low turbulence*

2. Out of focus blur:

   In the frequency domain, define $H(u,v)$ as the inverse DFT of

$$h(x,y) = \begin{cases} 1 & \text{if } x^2+y^2 \leq D_0^2 \\ 0 & \text{otherwise} \end{cases}$$

   In some situations, a simple model will be used to describe out of focus blur by letting:

$$H(u,v) = \begin{cases} 1 & \text{if } u^2+v^2 \leq D_0^2 \\ 0 & \text{otherwise} \end{cases}$$

   But this model is usually too simple and inaccurate.

3. Uniform Linear Motion Blur:

   Assume the image $f(x,y)$ undergoes planar motion during acquisition. Let $(x_0(t), y_0(t))$ be the motion components in the x- and y-directions. Denote the time by t and the duration of exposure by T.

   Then the observed image can be written as:

$$g(x,y) = \int_0^T f(x - x_0(t), y - y_0(t))\, dt$$

   We want to understand the motion blur in the frequency domain.

   Let $G(u,v) = DFT(g)(u,v)$, then:

$$\begin{aligned}
G(u,v) &= \sum_x \sum_y g(x,y) e^{-j\frac{2\pi}{N}(ux+vy)} \quad \text{(assume the image is a } N \times N \text{ image)} \\
&= \sum_x \sum_y \int_0^T f(x - x_0(t), y - y_0(t))\, dt\ e^{-j\frac{2\pi}{N}(ux+vy)} \\
&= \int_0^T \left[\sum_x \sum_y f(x - x_0(t), y - y_0(t)) e^{-j\frac{2\pi}{N}(ux+vy)} \right] dt
\end{aligned}$$

   Now, using the property that for

$$\tilde{f}(x,y) = f(x - x_0, y - y_0),$$

$$DFT(\tilde{f})(u,v) = DFT(f)(u,v) e^{-j\frac{2\pi}{N}(ux_0+vy_0)},$$

   we have

$$\begin{aligned}
G(u,v) &= \int_0^T \left[DFT(f)(u,v) e^{-j\frac{2\pi}{N}(ux_0(t)+vy_0(t))} \right] dt \\
&= DFT(f)(u,v) \underbrace{\int_0^T e^{-j\frac{2\pi}{N}(ux_0(t)+vy_0(t))}\, dt}_{H(u,v)},
\end{aligned}$$

   Therefore, degradation function in the frequency domain is given by:

$$H(u,v) = \int_0^T e^{-j\frac{2\pi}{N}(ux_0(t)+vy_0(t))}\, dt$$

2.3 Image deblurring algorithms in the frequency domain

A blurred image g can be modelled as: $g = h * f(x, y) + n(x, y)$
where h = blur function in the spatial domain;
$f(x, y)$ is the original image (clean) and
$n(x, y)$ is the noise.

In the frequency domain:

$$\begin{aligned} G(u, v) = \mathcal{F}(g)(u, v) &= c\mathcal{F}(h)(u, v)\mathcal{F}(f)(u, v) + \mathcal{F}(n)(u, v) \\ &= cH(u, v)F(u, v) + N(u, v) \end{aligned}$$

for some constant $c > 0$. By replacing H by $cH(u, v)$, we can ignore the constant c.

Deblurring methods (Suppose H is known)

Method 1: Direct inverse filtering

Let $T(u, v) = \dfrac{1}{H(u, v) + \varepsilon sgn(H(u, v))}$. Compute $\hat{F}(u, v) = G(u, v)T(u, v)$. Find inverse DFT of $\hat{F}(u, v)$ to get an image $\hat{f}(x, y)$.

(Here, $sgn(z) = 1$ if $Re(z) \geq 0$ and $sgn(z) = -1$ otherwise.)

Good: Simple.

Bad: $\hat{F}(u, v) = G(u, v)T(u, v) = F(u, v) + \dfrac{N(u, v)}{H(u, v) + \varepsilon sgn(H(u, v))}$

$H(u, v)$ is usually big for (u, v) close to $(0, 0)$ (associated to low frequency components) while small for (u, v) away from $(0, 0)$. Therefore, $\dfrac{N(u, v)}{H(u, v) + \varepsilon sgn(H(u, v))}$ is big (large gain in high frequency) and noise dominants.

Below is an illustration of how direct inverse filtering can boast up noises.

Original Image   Blurred Image   Restored with $H^{-1}(u, v)$

A small amount of noise saturates the inverse filter.

Method 2: Modified inverse filtering

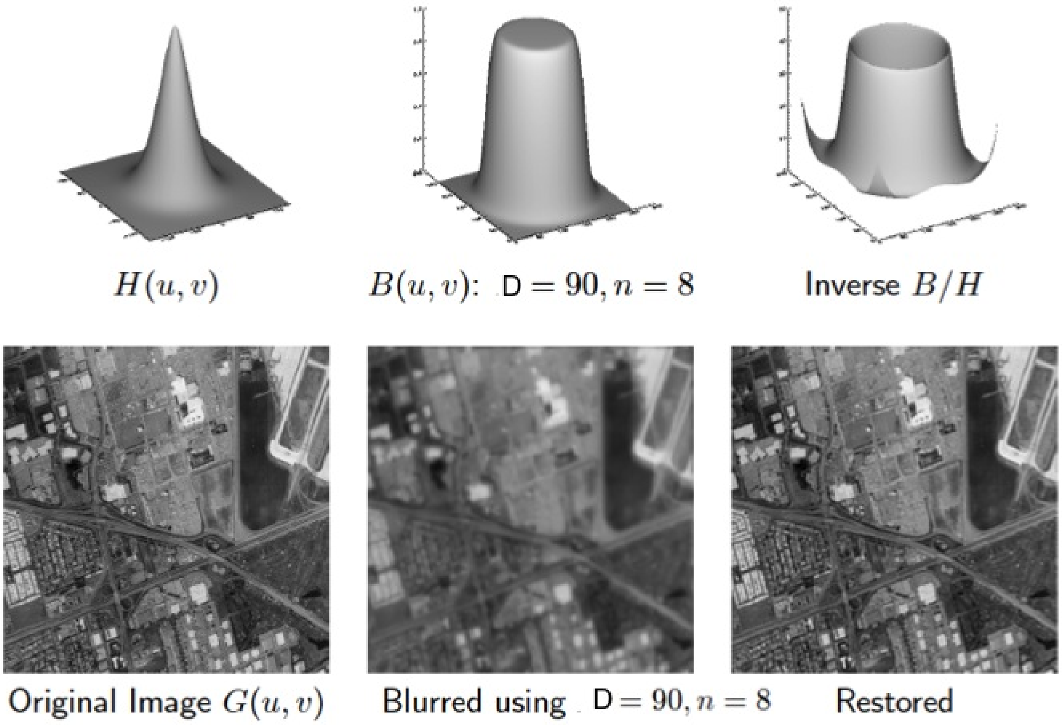
Let $B(u, v) = \dfrac{1}{1 + (\frac{u^2+v^2}{D^2})^n}$, and $T(u, v) = \dfrac{B(u, v)}{H(u, v) + \varepsilon sgn(H(u, v))}$, then

$$\hat{F}(u, v) = T(u, v)G(u, v) \approx F(u, v)B(u, v) + \frac{N(u, v)B(u, v)}{H(u, v) + \varepsilon sgn(H(u, v))}$$

$\dfrac{B(u, v)}{H(u, v) + \varepsilon sgn(H(u, v))}$ suppresses the high-frequency gain.

Bad: Has to choose D very carefully!!

Below is an illustration of how modified inverse filtering performs.



Method 3: Wiener Filter

The Wiener Filter is defined (in the frequency domain) as:

$$T(u,v) = \frac{\overline{H(u,v)}}{|H(u,v)|^2 + S_n(u,v)/S_f(u,v)}$$

where $S_n(u,v) = |N(u,v)|^2, S_f(u,v) = |F(u,v)|^2$ (Add parameters to avoid singularities)

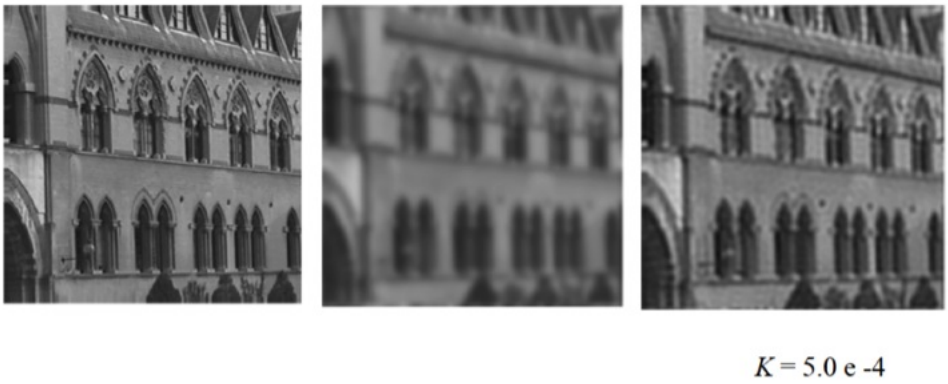
If $S_n(u,v)$ and $S_f(u,v)$ are not known, then we let $K = S_n(u,v)/S_f(u,v)$ to get

$$T(u,v) = \frac{\overline{H(u,v)}}{|H(u,v)|^2 + K}$$

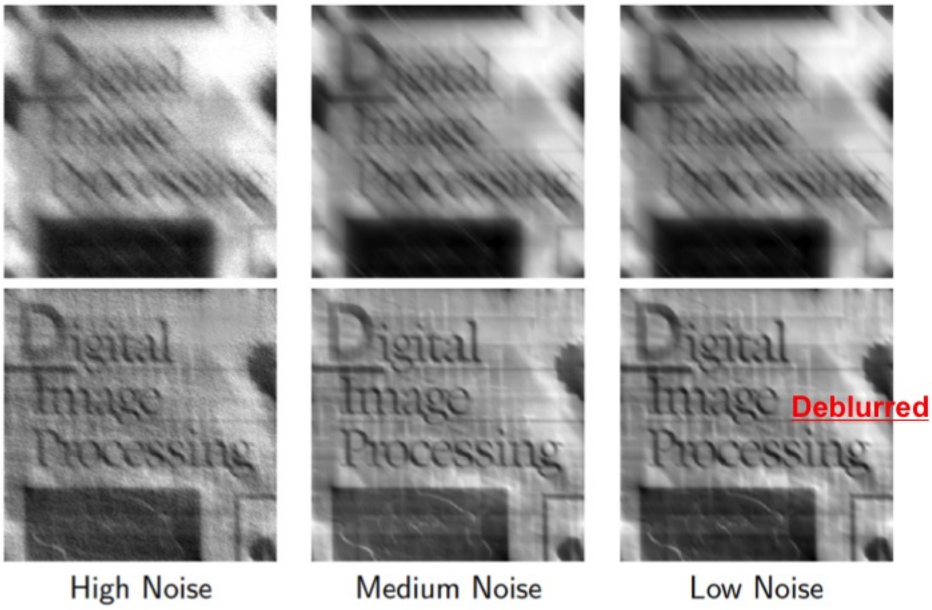
Hence, Wiener Filter can be described as the inverse filtering as follows:

$$\hat{F}(u,v) = \left[\underbrace{\left(\frac{1}{H(u,v)} \right)}_{\text{direct inverse filter}} \underbrace{\left(\frac{|H(u,v)|^2}{|H(u,v)|^2 + K} \right)}_{\text{modifier}} \right] G(u,v)$$

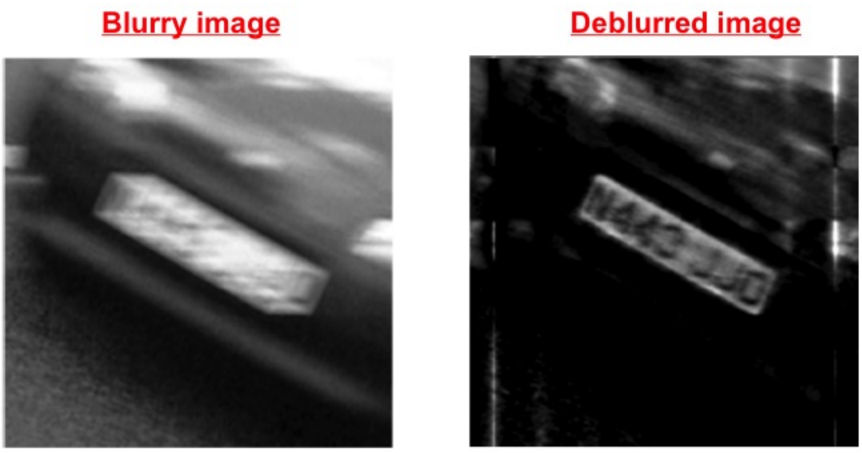
Below is an illustration of how Wiener filtering performs.



And below is an illustration of how Wiener filtering performs on noisy and blurry images.



Finally, we show how Wiener filtering performs for deblurring the car license plate.



What does Wiener filter do mathematically? (Optional, will not be covered)

We can show that (under certain condition), the Wiener filter minimizes the mean-square error (MSE).

(Sketch of proof)

We consider the continuous case to avoid the complicated indices. Let $g = h * f + n$, where h is the degradation function, n is the noise, f is the original clean image and g is our observed image.

Assume that f and n are spatially uncorrelated:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)n(x+r,y+s)\ dx\ dy = 0$$

for all r, s.

Then, we will show that the Wiener's filter minimizes the mean square error:

$$\mathcal{E}^2(\tilde{f}) = E((f-\tilde{f})^2) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [f(x,y) - \tilde{f}(x,y)]^2\ dx\ dy$$

where: $f(x,y)$ is the original image and $\tilde{f}(x,y) = \omega(x,y) * g(x,y)$ (where $g(x,y)$ is the observed image) is the Wiener filtered image.

Note that by Parserval's theorem:

$$\begin{aligned}\mathcal{E}^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (f(x,y) - \tilde{f}(x,y))^2\ dx\ dy \\ &= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(u,v) - \tilde{F}(u,v)|^2\ du\ dv \quad \text{(Parseval's theorem)}\end{aligned}$$

for some constant C, where $F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j(xu+yv)}dxdy$ and $\tilde{F}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(x,y)e^{-j(xu+yv)}dxdy$.

Let $G(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j(xu+yv)}dxdy$ and $N(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n(x,y)e^{-j(xu+yv)}dxdy$.

Then, we know: $\tilde{F} = WG = W(HF+N)$. In other words, $F - \tilde{F} = (1-WH)F - WN$ and

$$\mathcal{E}^2 = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |(1-WH)F - WN|^2\ du\ dv$$

Since f and n are spatially uncorrelated, we can show that:

$$\mathcal{E}^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |(1-WH)F|^2 + |WN|^2\ du\ dv.$$

We can regard $\mathcal{E}^2$ to be depending on W.

To minimize $\mathcal{E}^2(W)$, we consider:

$$\frac{d}{dt}|_{t=0}\mathcal{E}^2(W+tV) = 0 \text{ for all } V.$$

Hence, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -(1-\bar{W}\bar{H})H|F|^2V - (1-WH)\bar{H}|F|^2\bar{V} + \bar{W}|N|^2V + W|N|^2\bar{V} = 0$ for all V.

Put $V = -(1-WH)\bar{H}|F|^2 + W|N|^2$, we get:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|-(1-WH)\bar{H}|F|^2\bar{V} + W|N|^2\right|^2 = 0.$$

Thus,

$$2(-(1-\bar{W}\bar{H})H|F|^2 + \bar{W}|N|^2) = 0$$

$$\Leftrightarrow W = \frac{\bar{H}}{|H|^2 + |N|^2/|F|^2}.$$

Method 4: Constrained least square filtering

Drawback of Wiener filter:

1. $|N(u,v)|^2$ and $|F(u,v)|^2$ must be known.
2. Constant estimation of the ratio is not always suitable.

We consider a least square minimization model. The degradation process $g = h * f + n$ can be written in matrix form:

$$\vec{g} = D\vec{f} + \vec{n}$$

where $\vec{g} = \mathcal{S}(g), \vec{f} = \mathcal{S}(f), \vec{n} = \mathcal{S}(n)$, where $\mathcal{S}$ is the stacking operator.
Therefore, $\vec{g}, \vec{f}, \vec{n} \in \mathbb{R}^{MN}, D \in M_{MN \times MN}$.

Given $\vec{g}$, we want to find an estimate of f (or $\vec{f}$) such that it minimizes:

$$E(f) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} |\nabla^2 f(x,y)|^2$$

subject to the constraint that: $||\vec{g} - D\vec{f}||^2 = \epsilon$

In the discrete case, we can estimate $\nabla^2 f(x,y)$ by

$$\nabla^2 f(x,y) \approx f(x+1,y) + f(x,y+1) + f(x-1,y) + f(x,y-1) - 4f(x,y).$$

More generally, in the discrete case,

$$\nabla^2 f = p * f,$$

where $p = \begin{pmatrix} 0 & & \cdots & & 0 \\ & & 1 & & \\ \vdots & 1 & -4 & 1 & \vdots \\ & & 1 & & \\ 0 & & \cdots & & 0 \end{pmatrix}$.

Remark.

1. *The constraint means we want $||\vec{n}||^2 = ||\vec{g} - D\vec{f}||^2$ has a fixed level of noise. (Control the noise level + allow noise)*
2. *The energy $E(f)$ enhances the smoothness of f.*

Suppose $E(f)$ can be written as: $(L\vec{f})^T(L\vec{f})$ (L is the transformation matrix representing the convolution with p). Then, the constrained least square problem has the optimal solution in the spatial domain satisfies:

$$[D^T D + \gamma L^T L]\vec{f} = D^T \vec{g}$$

for some suitable parameter γ (which is related to the Lagrange multiplier).

In the frequency domain,

$$\hat{F}(u,v) := DFT(f)(u,v) = \frac{1}{N^2} \frac{\overline{H(u,v)}}{|H(u,v)|^2 + \gamma |P(u,v)|^2} G(u,v)$$

where $H(u,v) = DFT(h)(u,v)$, $G(u,v) = DFT(f)(u,v)$; $P(u,v) = DFT(p)(u,v)$ with

$$p(x,y) = \begin{pmatrix} 0 & & & \cdots & & & 0 \\ \vdots & & & \vdots & & & \vdots \\ 0 & \cdots & & 1 & & \cdots & 0 \\ 0 & \cdots & 1 & -4 & 1 & \cdots & 0 \\ 0 & \cdots & & 1 & & \cdots & 0 \\ \vdots & & & \vdots & & & \vdots \\ 0 & & & \cdots & & & 0 \end{pmatrix}$$

Sketch of proof:

From calculus, we know the minimizer must satisfy:

$$\mathcal{D} = \frac{\partial}{\partial \vec{f}}[\vec{f}^T L^T L \vec{f} + \lambda(\vec{g} - D\vec{f})^T(\vec{g} - D\vec{f})] = 0$$

where λ is the Lagrange multiplier. Here,

$$\frac{\partial K}{\partial \vec{f}} = \begin{pmatrix} \frac{\partial K}{\partial f_1} & \frac{\partial K}{\partial f_2} & \cdots & \frac{\partial K}{\partial f_{N^2}} \end{pmatrix}^T$$

Easy to check:

$$\frac{\partial \vec{f}^T \vec{a}}{\partial \vec{f}} = \vec{a}; \quad \frac{\partial \vec{b}^T \vec{f}}{\partial \vec{f}} = \vec{b}$$

Also, if A is an $N^2 \times N^2$ square matrix, then:

$$\frac{\partial \vec{f}^T A \vec{f}}{\partial \vec{f}} = (A + A^T)\vec{f}$$

$$\begin{aligned} \therefore \mathcal{D} = 0 &\Rightarrow (2L^T L)\vec{f} + \lambda(-D^T\vec{g} - D^T\vec{g} + 2D^T D\vec{f}) = 0 \\ &\Rightarrow (D^T D + \gamma L^T L)\vec{f} = D^T \vec{g} \end{aligned}$$

where $\gamma = \dfrac{1}{\lambda}$ and λ is the Lagrange multiplier.

Parameter γ can be determined by direct substitution into the equation:

$$[\vec{g} - D\vec{f}]^T[\vec{g} - D\vec{f}] = \epsilon$$

In the frequency domain, note that both D and L are block-circulant. Recall that a matrix A is **block-circulant** if:

$$A = \begin{pmatrix} A_0 & A_{N-1} & A_{N-2} & \cdots & A_1 \\ A_1 & A_0 & A_{N-1} & \cdots & A_2 \\ A_2 & A_1 & A_0 & \cdots & A_3 \\ \vdots & \vdots & \vdots & & \vdots \\ A_{N-1} & A_{N-2} & A_{N-3} & \cdots & A_0 \end{pmatrix}$$

where $A_0, A_1, \cdots, A_{N-1}$ are submatrices and they are themselves circulant matrices.

A matrix B is **circulant** if:

$$B = \begin{pmatrix} b_0 & b_{M-1} & b_{M-2} & \cdots & b_1 \\ b_1 & b_0 & b_{M-1} & \cdots & b_2 \\ b_2 & b_1 & b_0 & \cdots & b_3 \\ \vdots & \vdots & \vdots & & \vdots \\ b_{M-1} & b_{M-2} & b_{M-3} & \cdots & b_0 \end{pmatrix}$$

Remark. *Fact about circulant matrix:*

Let $B = \begin{pmatrix} b(0) & b(M-1) & \cdots & b(1) \\ b(1) & b(0) & \cdots & b(2) \\ \vdots & \vdots & \cdots & \vdots \\ b(M-1) & b(M-2) & \cdots & b(0) \end{pmatrix}$ *be a circulant matrix. Then the eigenvalues of* B *is given by:*

$$\lambda(k) = b(0) + b(1)e^{\frac{2\pi j}{M}(M-1)k} + b(2)e^{\frac{2\pi j}{M}(M-2)k} + \cdots + b(M-1)e^{\frac{2\pi j}{M}k}$$

where $k = 0, 1, 2, \cdots M-1$.

Its associated eigenvector is given by:

$$\vec{w}(k) = \begin{pmatrix} 1 \\ e^{\frac{2\pi j}{M}k} \\ e^{\frac{2\pi j}{M}2k} \\ \vdots \\ e^{\frac{2\pi j}{M}(M-1)k} \end{pmatrix}.$$

Recall that both D and L are block-circulant. In fact, D, L, D^T, L^T can be written as:

$$D = W\Lambda_D W^{-1}, D^T = W\Lambda_D^* W^{-1}, L = W\Lambda_L W^{-1}, L^T = W\Lambda_L^* W^{-1}$$

where W is invertible and Λ_D, Λ_L are diagonal matrices. In fact,

$$\Lambda_D(k,i) = \begin{cases} N^2\hat{D}\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

where $H = DFT(h)$. Similarly,

$$\Lambda_L(k,i) = \begin{cases} N^2\hat{L}\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

where $\hat{L} = DFT(p)$.

By substitution, $(D^TD + \gamma L^TL)\vec{f} = H^T\vec{g}$

$\Rightarrow \cancel{W}(\Lambda_D^*\Lambda_D + \gamma\Lambda_L^*\Lambda_L)W^{-1}\vec{f} = \cancel{W}\Lambda_D^*W^{-1}\vec{g}$

We can show that

$$\Lambda_D^*\Lambda_D = \begin{pmatrix} N^4|H(0,0)|^2 & & & & & \\ & N^4|H(1,0)|^2 & & & & \\ & & \ddots & & & \\ & & & N^4|H(N-1,0)|^2 & & \\ & & & & \ddots & \\ & & & & & N^4|H(N-1,N-1)|^2 \end{pmatrix}$$

and

$$\Lambda_L^*\Lambda_L = \begin{pmatrix} N^4|P(0,0)|^2 & & & & & \\ & N^4|P(1,0)|^2 & & & & \\ & & \ddots & & & \\ & & & N^4|P(N-1,0)|^2 & & \\ & & & & \ddots & \\ & & & & & N^4|P(N-1,N-1)|^2 \end{pmatrix}$$

Also, $W^{-1}\vec{f} = N\mathcal{S}(F), W^{-1}\vec{g} = N\mathcal{S}(G)$ where $F = DFT(f), G = DFT(g)$.

Combining all, we see that for all (u, v):

$$N^4[|\hat{H}(u,v)|^2 + \gamma|\hat{L}(u,v)|^2]NF(u,v) = N^2\overline{\hat{H}(u,v)}NG(u,v)$$

$$\Rightarrow N^2\frac{|\hat{H}(u,v)|^2 + \gamma|\hat{L}(u,v)|^2}{\overline{\hat{H}(u,v)}}F(u,v) = G(u,v)$$

Example 2.1. Consider a 3×3 image f. Let $\vec{f} = \mathcal{S}(f)$. Recall that the Laplacian of f can be calculated by:

$$\Delta = p * f \quad \text{where} \quad p = \begin{pmatrix} 0 & & \cdots & & 0 \\ & & 1 & & \\ \vdots & 1 & -4 & 1 & \vdots \\ & & 1 & & \\ 0 & & \cdots & & 0 \end{pmatrix}$$

Suppose $\mathcal{S}(\Delta f) = L\vec{f}$ for some matrix $L \in M_{9\times 9}$. Find L and show that L is block-circulant.

Solution. We can extend the image periodically as:

$$\begin{array}{c|ccc|c} f_{33} & f_{31} & f_{32} & f_{33} & f_{31} \\ \hline f_{13} & f_{11} & f_{12} & f_{13} & f_{11} \\ f_{23} & f_{21} & f_{22} & f_{23} & f_{21} \\ f_{33} & f_{31} & f_{32} & f_{33} & f_{31} \\ \hline f_{13} & f_{11} & f_{12} & f_{13} & f_{11} \end{array}$$

Then, L can be written as:

$$L = \underbrace{\left(\begin{array}{ccc|ccc|ccc} -4 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & -4 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -4 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & -4 \end{array}\right)}_{\text{block-circulant}} \begin{pmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{12} \\ f_{22} \\ f_{32} \\ f_{13} \\ f_{23} \\ f_{33} \end{pmatrix}$$

Example 2.2. Consider a 2×2 image f. Let $g = h * f$ where $h \in M_{2\times 2}$. Let $H \in M_{4\times 4}$ such that $H\vec{f} = \vec{g}$ where $\vec{f} = \mathcal{S}(f), \vec{g} = \mathcal{S}(g)$. What is H?

Solution. $g_{mn} = \sum_{i=0}^{1}\sum_{j=0}^{1} h(m-i, n-j)f_{ij}$

$$\therefore \begin{pmatrix} g_{00} \\ g_{10} \\ g_{01} \\ g_{11} \end{pmatrix} = \underbrace{\left(\begin{array}{c|c|c|c} h_{0,0} & h_{-1,0} & h_{0,-1} & h_{-1,-1} \\ h_{1,0} & h_{0,0} & h_{1,-1} & h_{0,-1} \\ h_{0,1} & h_{-1,1} & h_{0,0} & h_{-1,0} \\ h_{1,1} & h_{0,1} & h_{1,0} & h_{0,0} \end{array}\right)}_{H} \begin{pmatrix} f_{00} \\ f_{10} \\ f_{01} \\ f_{11} \end{pmatrix}$$

Let

$$H_u = \begin{pmatrix} h(0,u) & h(-1,u) \\ h(1,u) & h(0,u) \end{pmatrix} = \begin{pmatrix} h(0,u) & h(1,u) \\ h(1,u) & h(0,u) \end{pmatrix} \quad \text{by periodic condition}$$

then, it is easy to check

$$H = \begin{pmatrix} H_0 & H_{-1} \\ H_1 & H_0 \end{pmatrix} = \begin{pmatrix} H_0 & H_1 \\ H_1 & H_0 \end{pmatrix} \quad \text{by periodic condition}$$

Remark. *In general, suppose f is an $N \times N$ image, $g = h * f$ and $H\vec{f} = \vec{g}$. Then,*

$$H = \begin{pmatrix} H_0 & H_{-1} & H_{-2} & \cdots & H_{-N+1} \\ H_1 & H_0 & H_{-1} & \cdots & H_{-N+2} \\ H_2 & H_1 & H_0 & \cdots & H_{-N+3} \\ \vdots & \vdots & \vdots & & \vdots \\ H_{N-1} & H_{N-2} & H_{N-3} & \cdots & H_0 \end{pmatrix}$$

where

$$H_u = \begin{pmatrix} h(0,u) & h(N-1,u) & h(N-2,u) & \cdots & h(1,u) \\ h(1,u) & h(0,u) & h(N-1,u) & \cdots & h(2,u) \\ h(2,u) & h(1,u) & h(0,u) & \cdots & h(3,u) \\ \vdots & \vdots & \vdots & & \vdots \\ h(N-1,u) & h(N-2,u) & h(N-3,u) & \cdots & h(0,u) \end{pmatrix}$$

Easy to see that H is block circulant and H_u is circulant.

<u>Diagonalization of H</u>

Let H be the block-circulant matrix as defined above. Define a matrix with elements:

$$W_N(k,n) := \frac{1}{\sqrt{N}} exp\left(\frac{2\pi j}{N} kn\right) \qquad 0 \le n \le N-1$$

Consider the **Kronecker product** $\otimes$ of W_N with itself:

$$W := W_N \otimes W_N$$

The **Kronecker product** of two matrices are given by:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1N}B \\ a_{21}B & a_{22}B & \cdots & a_{2N}B \\ \vdots & \vdots & & \vdots \\ a_{N1}B & a_{N2}B & \cdots & a_{NN}B \end{pmatrix}$$

Easy to check: $W^{-1} = W_N^{-1} \otimes W_N^{-1}$ where:

$$W_N^{-1}(k,n) := \frac{1}{\sqrt{N}} exp\left(-\frac{2\pi j}{N} kn\right) \qquad 0 \le n \le N-1$$

Let

$$\Lambda(k,i) = \begin{cases} N^2 \hat{H}\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

where $\hat{H}$ = DFT of the point spread function h, $\left\lfloor \frac{k}{N} \right\rfloor$ = largest integer smaller than or equal to $\frac{k}{N}$ and $\mathrm{mod}_N(k) = k(\mathrm{mod}\ N)$ (e.g. $10(\mathrm{mod}\ 3) = 1$)

Then, we can show that $H = W\Lambda W^{-1}$ and $H^{-1} = W\Lambda^{-1}W^{-1}$.

Also, $H^T = W\Lambda^* W^{-1}$. (Λ^* is the complex conjugate of Λ)

By direct calculation, it is easy to check that $W^{-1}\vec{g} = N\mathcal{S}(G)$ where $G = DFT(g)$.

Example 2.3. Assume that:

$$G = \begin{pmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{pmatrix} \quad \text{and} \quad W_3^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & exp\left(-\frac{2\pi j}{3}\right) & exp\left(-\frac{2\pi j}{3}2\right) \\ 1 & exp\left(-\frac{2\pi j}{3}2\right) & exp\left(-\frac{2\pi j}{3}\right) \end{pmatrix}$$

Then:

$$W^{-1} = W_3^{-1} \otimes W_3^{-1}$$

$$= \frac{1}{3} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}4} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}3} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2}
\end{pmatrix}$$

Note that $e^{-\frac{2\pi j}{3}3} = e^{-2\pi j} = 1$, and $e^{-\frac{2\pi j}{3}4} = e^{-\frac{2\pi j}{3}3} e^{-\frac{2\pi j}{3}} = e^{-\frac{2\pi j}{3}}$, then

$$W^{-1}g = \frac{1}{3} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2}
\end{pmatrix}
\begin{pmatrix} g_{00} \\ g_{10} \\ g_{20} \\ g_{01} \\ g_{11} \\ g_{21} \\ g_{02} \\ g_{12} \\ g_{22} \end{pmatrix}$$

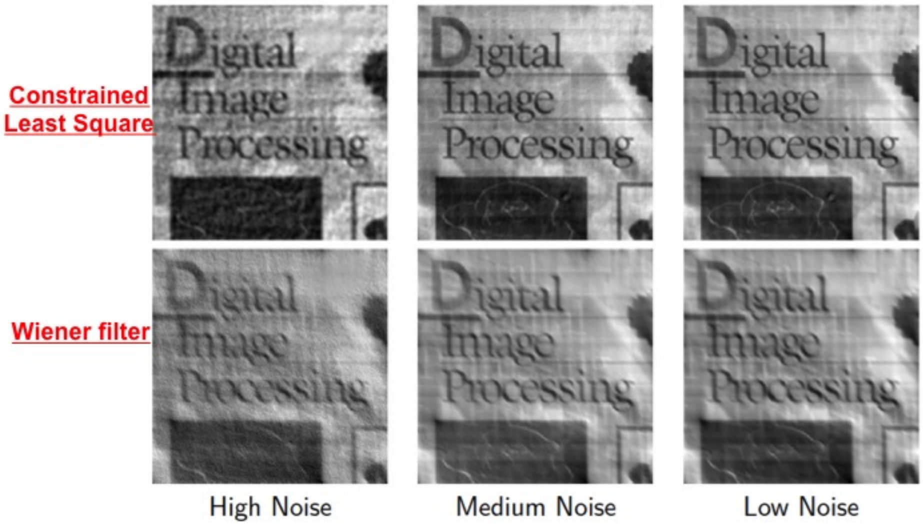
$$= \frac{1}{3} \begin{pmatrix}
g_{00} + g_{10} + g_{20} + g_{01} + g_{11} + g_{21} + g_{02} + g_{12} + g_{22} \\
g_{00} + g_{10}e^{-\frac{2\pi j}{3}} + g_{20}e^{-\frac{2\pi j}{3}2} + g_{01} + g_{11}e^{-\frac{2\pi j}{3}} + g_{21}e^{-\frac{2\pi j}{3}2} + g_{02} + g_{12}e^{-\frac{2\pi j}{3}} + g_{22}e^{-\frac{2\pi j}{3}2} \\
\vdots \\
\vdots \\
\vdots
\end{pmatrix}$$

Careful examination of the elements of this vector shows that they are the Fourier components of G, multiplied with 3, compared at various combinations of frequencies (u, v), for $u, v = 0, 1, 2$, and arranged as follows:

$$3 \times \begin{pmatrix} \hat{G}(0,0) \\ \hat{G}(1,0) \\ \hat{G}(2,0) \\ \hat{G}(0,1) \\ \hat{G}(1,1) \\ \hat{G}(2,1) \\ \hat{G}(0,2) \\ \hat{G}(1,2) \\ \hat{G}(2,2) \end{pmatrix}$$

Note that $\hat{G}(m,n) = \frac{1}{9} \sum g_{kl} e^{-\frac{2\pi j}{3}(mk+nl)} \Rightarrow \hat{G}(1,0) = \frac{1}{9} \sum g_{kl} e^{-\frac{2\pi j}{3}(k)}$. This shows that $W^{-1}g$ yields N times the Fourier transform of G, as a column vector.

Below is an illustration of how Constrained least square filtering performs on noisy and blurry images (compared with the Wiener's filtering).



3 Image sharpening in the frequency domain

The goal of image sharpening is to enhance an image so that it shows more obvious edges.

Method 1: Laplacian mask

Recall: $\Delta f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$.

By Taylor's expansion:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &\approx \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2} \\
\frac{\partial^2 f}{\partial y^2} &\approx \frac{f(x,y+h) - 2f(x,y) + f(x,y-h)}{h^2} \\
\therefore \Delta f(x,y) &\approx \frac{f(x,y+h) + f(x,y-h) + f(x+h,y) + f(x-h,y) - 4f(x,y)}{h^2}
\end{aligned}$$

In the case of images, we let $h = 1$. Hence, $\Delta f(x,y) = l * f(x,y)$ for some matrix l.

In the frequency domain,

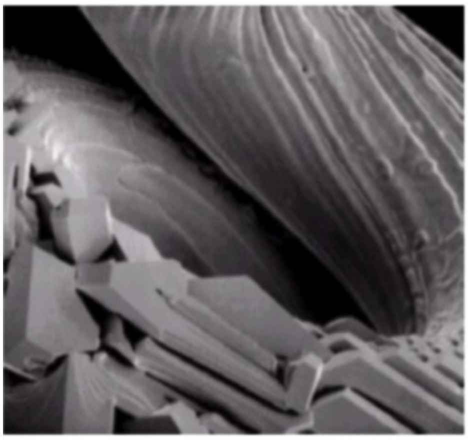
$$\begin{aligned}
DFT(g) &= DFT(f) - DFT(\Delta f) \\
&= DFT(F)(u,v) + DFT(l)DFT(F)(u,v) \\
&= (1 - H_{laplace}(u,v))F(u,v)
\end{aligned}$$

where $H_{laplace}(u,v) = DFT(l)(u,v)$

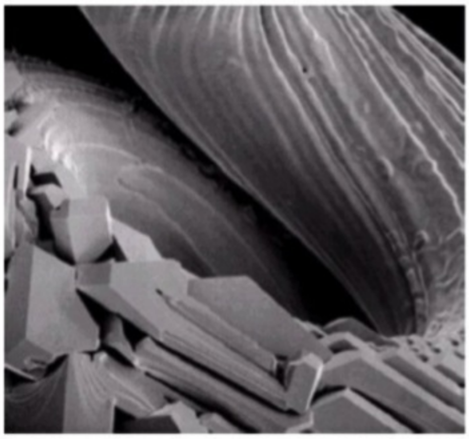
Below is an illustration of how Laplacian masking performs on two different images:

Original image

Laplacian mask



Original image



Laplacian mask

<u>Method 2:</u> Unsharp masking

<u>Idea:</u> Image sharpening = Add back high frequency component.

Definition 3.1. Let f be an input image (may be blurry). Compute smoother image (by Gaussian filter or mean filter) f_{smooth}. Define the sharper image g as:

$$g(x, y) = f(x, y) + k(f(x, y) - f_{smooth}(x, y))$$

When $k = 1$, the method is called <u>unsharp masking</u>.
When $k > 1$, the method is called <u>highboost filtering</u>.

In the frequency domain, let

$$DFT(f_{smooth})(u, v) = H_{LP}(u, v)DFT(f)(u, v),$$

where H_{LP} is the low pass filter.

Then: $DFT(g)(u, v) = [1 + k(1 - H_{LP}(u, v))]DFT(f)(u, v)$.

Below is an illustration of how unsharp masking performs:

(a) Original

(b) Global, Gaussian 121×121