Math 3360: Mathematical Imaging

Assignment 2

Due: October 6 before 1159PM

Please give reasons in your solutions.

1. (a) Let $A, B \in M_{4\times 4}(\mathbb{R})$ and the image transformation $\mathcal{O} : M_{4\times 4}(\mathbb{R}) \to M_{4\times 4}(\mathbb{R})$ is defined by:
$$\mathcal{O}(f) = AfB,$$
please show that the transformation matrix H of $\mathcal{O}$ is given by:
$$H = B^T \otimes A.$$
   (b) In more general cases, let $A, B \in M_{n\times n}(\mathbb{R})$ and the image transformation $\mathcal{O} : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ is defined by:
$$\mathcal{O}(f) = AfB,$$
please show that the transformation matrix H of $\mathcal{O}$ is also given by:
$$H = B^T \otimes A.$$

2. Compute the singular value decomposition(SVD) of
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$
Please show all your steps in detail.

3. Let $H_n(t)$ be the n-th Haar function, where $n \in \mathbb{N} \cup \{0\}$.
   (a) Give the definition of $H_n(t)$.
   (b) Write down the Haar transformation matrix $\tilde{H}$ for 4×4 images.
   (c) Suppose $A = \begin{pmatrix} 5 & 4 & 6 & 6 \\ 6 & 1 & 6 & 3 \\ 1 & 2 & 1 & 5 \\ 6 & 4 & 6 & 1 \end{pmatrix}$. Compute the Haar transform A_{Haar} of A, and compute the reconstructed image $\tilde{A}$ after setting the two smallest (in absolute value) nonzero entries of A_{Haar} to 0.

4. For an $n \times n$ image g of real entries, let $g = UfV^T$, where $U, V, f \in M_{n\times n}(\mathbb{R})$.
   (a) Show that
$$g = \sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}\vec{u_i}\vec{v_j}^T,$$
where
$$U = \begin{pmatrix} \vec{u_1} & \vec{u_2} & \cdots & \vec{u_n} \end{pmatrix}, V^T = \begin{pmatrix} \vec{v_1}^T \\ \vec{v_2}^T \\ \vdots \\ \vec{v_n}^T \end{pmatrix},$$
   (b) Show that if f is diagonal, then the trace of g is given by
$$\text{tr}(g) = \sum_{k=1}^{n} g_{kk} = \sum_{k=1}^{n}\sum_{l=1}^{n} f_{ll}u_{kl}v_{kl}.$$

5. Suppose $J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, Compute $\hat{J} = DFT(J)$.

6. **(Optimal) Programming exercise:** Compress a digital image using SVD, please try to show the rank-k approximations with $k = 5, 10, 50$ respectively.

   Hint: You can use any programming language (python, matlab, R and so on) with any third-party library, you DON'T need to implement the SVD algorithm yourself. Please submit the following as your solutions:

   1. your code,
   2. original image,
   3. rank-k approximations for $k = 5, 10, 50$.