Math 3360: Mathematical Imaging

Assignment 1

Due: September 22, 2023

Please give reasons in your solutions.

1. Prove or disprove if the following image transformation $\mathcal{O} : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ is linear.

   (a) Let $a \in \mathbb{R}$, $A \in M_{N\times N}(\mathbf{R})$. For any $f \in M_{N\times N}(\mathbf{R})$, $\mathcal{O}(f) = af + Af^T$, where f^T is the transpose of f.

   (b) Let $A \in M_{N\times N}(\mathbf{R})$. For any $f \in M_{N\times N}(\mathbf{R})$, $\mathcal{O}(f) = fAf$.

   (c) Let $k \in M_{N\times N}(\mathbf{R})$. For any $f \in M_{N\times N}(\mathbf{R})$, $\mathcal{O}(f) = k * f$, where $*$ denote the discrete convolution.

2. Let $A = (a_{ij})_{1\le i,j\le 2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = (b_{ij})_{1\le i,j\le 2} = \begin{pmatrix} 1 & 3 \\ 5 & 6 \end{pmatrix}$. Define the image transformation $\mathcal{O} = M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by $\mathcal{O}(f) = AfB$. Let

$$H^{1,2} = \begin{pmatrix} h^{1,2}(1,1) & h^{1,2}(1,2) \\ h^{1,2}(2,1) & h^{1,2}(2,2) \end{pmatrix},$$

   where $h^{\alpha,\beta}(x,y)$ is the point spread function of $\mathcal{O}$. Compute $H^{1,2}$.

3. Let $f = (f_{ij})_{1\le i,j\le 3} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = (b_{ij})_{1\le i,j\le 3} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & 4 \end{pmatrix}$. f and B are periodically extended.

   (a) Compute $f * B$, where $*$ denote the discrete convolution.

   (b) Let $g = f * B \in M_{3\times 3}(\mathbb{R})$, show that for all $1 \le \alpha, \beta \le 3$

$$g(\alpha,\beta) = 4f_{\alpha,\beta} - f_{\alpha+1,\beta} - f_{\alpha-1,\beta} - f_{\alpha,\beta+1} - f_{\alpha,\beta-1},$$

   where $g(\alpha,\beta)$ are the α-th row, β-th column of g.

4. Define a linear image transformation $\mathcal{O} : M_{N\times N}(\mathbf{R}) \to M_{N\times N}(\mathbf{R})$ by

$$\mathcal{O}(f)(\alpha,\beta) = \frac{f(\alpha+1,\beta) + 2f(\alpha-1,\beta) + 3f(\alpha,\beta+1) + f(\alpha,\beta-1) - 8f(\alpha,\beta)}{4}.$$

   Show that $\mathcal{O}(f) = k * f$ for some $k \in M_{N\times N}(\mathbf{R})$ and find this k.

5. Compute the singular value decomposition (SVD) of

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   Please show all your steps in detail.