Math 3360: Mathematical Imaging

Assignment 5

Due: November 29 before 1159PM

Please give reasons in your solutions.

1. Consider a periodically extended 4×4 image $I = (I(x,y))_{0 \le x,y \le 3}$ given by:

$$I = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & b & 0 & 1 \\ a & 0 & 0 & 0 \\ 0 & 0 & c & 1 \end{pmatrix}$$

   Given that the discrete Laplacian ΔI of I is given by the formula:

$$\Delta I(x,y) = -4I(x,y) + I(x+1,y) + I(x-1,y) + I(x,y+1) + I(x,y-1) \text{ for } 0 \le x,y \le 3.$$

   We perform the Laplacian masking on I to get a sharpen image I_{sharp}. Suppose I_{sharp} is given by

$$I_{sharp} = \begin{pmatrix} 0 & 0 & 5 & -3 \\ -1 & -5 & -1 & 5 \\ 5 & 0 & 0 & -3 \\ -2 & 0 & -2 & 5 \end{pmatrix}.$$

   Find a, b and c. (**Hint**: You may want to use the formula of Laplacian masking in the spatial domain: $I_{sharp} = I - \Delta I$.)

2. Consider a 4×4 periodically extended image $I = (I(k,l))_{0 \le k,l \le 3}$ given by:

$$I = \begin{pmatrix} a & 0 & c & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

   where a, b and c are positive numbers.

   We apply the 3×3 mean filter to I to obtain I^{mean}. Suppose $I^{mean}(0,3) = 1$, $I^{mean}(1,2) = 8/9$ and $I^{mean}(1,0) = 5/9$. Find a, b and c.

3. Consider the following periodically extended 4×4 image $f = (f(x,y))_{0 \le x,y \le 3}$:

$$f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

   (a) Consider the ideal low-pass filter $H_{LP} = (H_{LP}(x,y))_{0 \le x,y \le 3}$ of radius 2. Explain with details why H_{LP} is given by:

$$I = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

   (b) We apply the ideal low-pass filter of radius 2, H_{LP} to perform unsharp masking on f to get f_{sharp}. Find f_{sharp}. Please show all your steps.

4. Apply the following filters:
the 3×3 mean filter;
the 3×3 median filter;
convolution filter $\frac{1}{12}\begin{pmatrix} 6 & 2 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ where the indices are starting from 0 , on the following periodically extended 4×4 images:

(a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & 3 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}$

5. Consider an image denoising model to find $f : [a, b] \times [a, b] \to \mathbb{R}$ that minimizes:

$$E(f) = \int_a^b \int_a^b (f(x, y) - g(x, y))^2 + \int_a^b \int_a^b K(x, y)|\nabla f(x, y)|^2 dxdy$$

Assuming $f(x, y) = g(x, y) = 0$ for (x, y) on the boundary of $[a, b] \times [a, b]$. Suppose f minimizes $E(f)$. Show that f satisfies:

$$f(x, y) - g(x, y) - \nabla \cdot (K(x, y)\nabla f(x, y)) = 0 \text{ in } [a, b] \times [a, b]$$