MATH3360 Mathematical Image Processing
Final Practice

1. The Butterworth high-pass filter H with radius D_0 and order n is defined as

$$H(u,v) = \frac{1}{1 + (D_0/D(u,v))^n},$$

where $D(u,v) = u^2 + v^2$. Given an image $I = (I(m,n))_{0 \le m,n \le 2N}$ and $N > 100$, apply Butterworth high-pass filter on $DFT(I) = (\hat{I}(u,v))_{0 \le u,v \le 2N}$ then get $G(u,v)$. Suppose

$$G(3,4) = \frac{1}{2}\hat{I}(3,4) \text{ and } G(2N-6,8) = \frac{16}{17}\hat{I}(2N-6,8),$$

where $\hat{I}(3,4) \neq 0$ and $\hat{I}(2N-6,8) \neq 0$. Find D_0 and n.

2. Consider a 4×4 periodically extended image $I = (I(k,l))_{0 \le k,l \le 3}$ given by:

$$I = \begin{pmatrix} a & b & a & b \\ b & a & b & a \\ a & b & a & b \\ b & a & b & a \end{pmatrix},$$

where a and b are distinct positive numbers.

Let $h = (h(k,l))_{0 \le k,l \le 3} = \frac{1}{8}\begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, which is periodically extended.

The Gaussian low-pass filter H of variance σ^2 is defined by:

$$H(u,v) = exp\left(-\frac{u^2+v^2}{\sigma^2}\right).$$

Let $I_1(u,v) = H_1(u,v)DFT(I)(u,v)$, where the H_1 is the Gaussian low-pass filter of variance ab. Suppose $I_1(2,2) = e^{-\frac{1}{4}}DFT(I)(2,2)$ and $h * f(2,1) = 6$. Find a and b.

3. Compute the degradation functions in the frequency domain that correspond to the following $M \times N$ convolution kernels h, i.e. find $H \in M_{M\times N}(\mathbb{C})$ such that

$$DFT(h * f)(u,v) = H(u,v)DFT(f)(u,v)$$

for any periodically extended $f \in M_{M\times N}(\mathbb{R})$:

(a) Assuming integer k satisfies $k \le \min\{\frac{M}{2}, \frac{N}{2}\}$,

$$h_1(x,y) = \begin{cases} \frac{1}{(2k+1)^2} & \text{if } dist(x, M\mathbb{Z}) \le k \text{ and } dist(y, N\mathbb{Z}) \le k, \\ 0 & \text{otherwise;} \end{cases}$$

(b) Letting $r > 1$,

$$h_2(x,y) = \begin{cases} \frac{r}{r+4} & \text{if } D(x,y) = 0, \\ \frac{1}{r+4} & \text{if } D(x,y) = 1, \\ 0 & \text{otherwise;} \end{cases}$$

(c)

$$h_3(x,y) = \begin{cases} \frac{1}{4} & \text{if } D(x,y) = 0, \\ \frac{1}{8} & \text{if } D(x,y) = 1, \\ \frac{1}{16} & \text{if } D(x,y) = 2, \\ 0 & \text{otherwise;} \end{cases}$$

(d)

$$h_4(x,y) = \begin{cases} -4 & \text{if } D(x,y) = 0, \\ 1 & \text{if } D(x,y) = 1, \\ 0 & \text{otherwise;} \end{cases}$$

(e) Letting $a, b \in \mathbb{Z}$ and $T \in \mathbb{N} \setminus \{0\}$ such that $|a|(T-1) < M$ and $|b|(T-1) < N$,

$$h_5(x,y) = \begin{cases} \frac{1}{T} & \text{if } (x,y) \in \{(at, bt) : t = 0, 1, \cdots, T-1\} \\ 0 & \text{otherwise.} \end{cases}$$

4. For any periodically extended $N \times N$ image f, define

$$G_x(f)(x,y) = \frac{1}{4}f(x+1,y) + \frac{1}{2}f(x,y) + \frac{1}{4}f(x-1,y)$$
$$\text{and } G_y(f)(x,y) = \frac{1}{4}f(x,y+1) + \frac{1}{2}f(x,y) + \frac{1}{4}f(x,y-1).$$

(a) Find an $N \times N$ image h such that for any periodically extended $N \times N$ image f,

$$h * f = G_x \circ G_y(f).$$

(b) Let $H(u,v)$ be the LPF such that

$$DFT(h * f)(u,v) = H(u,v)DFT(f)(u,v),$$

where h is the convolution kernel from (a). Using H, perform unsharp masking (i.e. $k = 1$) on the following periodically extended 4×4 image

$$f = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

5. Let $W_N(n,k) = \dfrac{1}{\sqrt{N}} e^{2\pi j \frac{nk}{N}}$ for $0 \le n, k \le N-1$ and $W = W_N \otimes W_N$. Suppose $N = 4$, we have following problems:

(a) Prove that $W^{-1} = \overline{W_N} \otimes \overline{W_N}$.

(b) Show that $W^{-1}\mathcal{S}(f) = N\mathcal{S}(\hat{f})$ for any $f \in M_{N\times N}(\mathbb{C})$, where $\hat{f} = DFT(f)$.

6. Let $f = (f_{ij})_{0 \le i,j \le N-1} \in M_{N\times N}(\mathbb{R})$ be a clean image. Suppose f is blurred to g under a motion which is given by:

$$g(x,y) = \sum_{t=0}^{3} f(x+t, y).$$

Show that $DFT(g)(u,v) = H(u,v)DFT(f)(u,v)$ and find $H(u,v)$.

7. Given $N^2 \times N^2$ block-circulant real matrices D and L, $N \times N$ image g and fixed parameter $\varepsilon > 0$, the constrained least square filtering aims to find $f \in M_{N\times N}$ that minimizes:

$$E(f) = [L\mathcal{S}(f)]^T[L\mathcal{S}(f)]$$

subject to the constraint:

$$[\mathcal{S}(g) - D\mathcal{S}(f)]^T[\mathcal{S}(g) - D\mathcal{S}(f)] = \varepsilon,$$

where $\mathcal{S}$ is the stacking operator.

Given that the optimal solution f that solves the constrained least square problem satisfies $[\lambda D^T D + L^T L]\mathcal{S}(f) = \lambda D^T \mathcal{S}(g)$ for some parameter λ. Find $DFT(f)$ in terms of $DFT(g)$, $DFT(h)$, $DFT(p)$ and λ, where $L\mathcal{S}(\varphi) = \mathcal{S}(p * \varphi)$ for any $\varphi \in M_{N\times N}(\mathbb{R})$.

8. Given $g \in M_{N\times N}(\mathbb{R})$, block circulant $D, L_1, L_2 \in M_{N^2\times N^2}(\mathbb{R})$ and $\varepsilon > 0$, we aim to minimize $\|L_1\vec{f}\|_2^2 + \|L_2\vec{f}\|_2^2$ subject to $\|\vec{g} - D\vec{f}\|_2^2 = \varepsilon$ over $f \in M_{N\times N}(\mathbb{R})$, where $\vec{f} = \mathcal{S}(f)$ and $\vec{g} = \mathcal{S}(g)$ vectorized by the stack operator $\mathcal{S}$.

Given Lagrange multiplier λ for the equality constraint, show that if f is a minimizer of the above constrained minimization problem, then

$$(\lambda D^T D + L_1^T L_1 + L_2^T L_2)\vec{f} = \lambda D^T \vec{g}.$$

Please prove your answer with details.

9. Given a 2D simple connected domain D and a noisy image $I : D \to \mathbb{R}$, we consider the following image denoising model to restore the original clean image $f : D \to \mathbb{R}$ that minimizes:

$$E(f) = \int_D (f(x,y) - I(x,y))^2 \, dxdy + \int_D \sqrt{|\nabla f(x,y)|^2 + \epsilon} \, dxdy$$

where small parameter $\epsilon > 0$.

(a) If f minimizes $E(f)$, show that f could satisfy the following conditions:

$$\begin{cases} 2f(x,y) - 2I(x,y) - \nabla \cdot \left(\dfrac{\nabla f(x,y)}{\sqrt{|\nabla f(x,y)|^2 + \epsilon}} \right) = 0 \text{ for } (x,y) \in D, \\ \langle \dfrac{\nabla f(x,y)}{\sqrt{|\nabla f(x,y)|^2 + \epsilon}}, \vec{n} \rangle = 0 \text{ for } (x,y) \in \partial D. \end{cases}$$

where $\vec{n}$ is the outward normal vector on ∂D.

(b) Derive an iterative scheme, which updates f_n to f_{n+1} with time step $\tau > 0$, to minimize $E(f)$.

10. Given a noisy image $I : D \to \mathbb{R}$, we consider the following image denoising model to restore the original clean image $f : D \to \mathbb{R}$ that minimizes:

$$E(f) = \int_D (f(x,y) - I(x,y))^2 \, dxdy + \int_D |\nabla f(x,y)|^2 \, dxdy.$$

where $|\nabla f(x,y)|^2 = (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2$.

(a) Derive an iterative scheme, which updates f^n to f^{n+1} with time step $\tau > 0$, to minimize $E(f)$. This is the same model as discussed in the lectures. Please show all your steps with details, including detailed explanations on why E is iteratively decreasing. Missing detailed steps will result in mark deductions.

(b) If E is modified to $\widetilde{E}$ defined as follows:

$$\widetilde{E}(f) = \int_D \sqrt{(f(x,y) - I(x,y))^2 + \epsilon^2} \, dxdy + \int_D \sqrt{|\nabla f(x,y)|^2 + \epsilon^2} \, dxdy,$$

where $\epsilon > 0$ is a small parameter bigger than 0. Derive an iterative scheme, which updates f^n to f^{n+1} with time step $\tau > 0$, to minimize $\widetilde{E}(f)$. Please show all your steps with details, including detailed explanations on why $\widetilde{E}$ is iteratively decreasing. Missing detailed steps will result in mark deductions.