Math 3360: Mathematical Imaging

Assignment 5

1. Consider a periodically extended 4×4 image $I = (I(x,y))_{0 \le x,y \le 3}$ given by:

$$I = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & b & 0 & 1 \\ a & 0 & 0 & 0 \\ 0 & 0 & c & 1 \end{pmatrix}$$

Given that the discrete Laplacian ΔI of I is given by the formula:

$$\Delta I(x,y) = -4I(x,y) + I(x+1,y) + I(x-1,y) + I(x,y+1) + I(x,y-1) \text{ for } 0 \le x,y \le 3.$$

We perform the Laplacian masking on I to get a sharpen image I_{sharp}. Suppose I_{sharp} is given by

$$I_{sharp} = \begin{pmatrix} 0 & 0 & 5 & -3 \\ -1 & -5 & -1 & 5 \\ 5 & 0 & 0 & -3 \\ -2 & 0 & -2 & 5 \end{pmatrix}.$$

Find a, b and c. (**Hint**: You may want to use the formula of Laplacian masking in the spatial domain: $I_{sharp} = I - \Delta I$.)

Solution: We have

$$\Delta I = \begin{pmatrix} 0 & b+1 & c-4 & 3 \\ a+b+1 & -4b & b+2 & -4 \\ -4a & a+b & c & a+2 \\ a+1 & c & 2-4c & c-4 \end{pmatrix}$$

and then

$$I_{sharp} = I - \Delta I = \begin{pmatrix} 0 & -b-1 & -c+5 & -3 \\ -a-b-1 & 5b & -b-2 & 5 \\ 5a & -a-b & -c & -a-2 \\ -a-1 & -c & 5c-2 & 5-c \end{pmatrix}.$$

Therefore, it's clear that $a = 1$, $b = -1$ and $c = 0$.

2. Consider a 4×4 periodically extended image $I = (I(k,l))_{0 \le k,l \le 3}$ given by:

$$I = \begin{pmatrix} a & 0 & c & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where a, b and c are positive numbers.

We apply the 3×3 mean filter to I to obtain I^{mean}. Suppose $I^{mean}(0,3) = 1$, $I^{mean}(1,2) = 8/9$ and $I^{mean}(1,0) = 5/9$. Find a, b and c.

Solution: The information implies that

$$I^{mean}(0,3) = \frac{1}{9}(a+c) = 1,$$
$$I^{mean}(1,2) = \frac{1}{9}(b+c) = \frac{8}{9},$$
$$I^{mean}(1,0) = \frac{1}{9}(a+b) = \frac{5}{9}.$$

Therefore, we get $a = 3$, $b = 2$ and $c = 6$.

3. Consider the following periodically extended 4×4 image $f = (f(x,y))_{0 \le x,y \le 3}$:

$$f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

(a) Consider the ideal low-pass filter $H_{LP} = (H_{LP}(x,y))_{0 \le x,y \le 3}$ of radius 2. Explain with details why H_{LP} is given by:

$$H_{LP} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

(b) We apply the ideal low-pass filter of radius 2, H_{LP} to perform unsharp masking on f to get f_{sharp}. Find f_{sharp}. Please show all your steps.

Solution:

(a) Since the image f is taken from $0 \le x, y \le 3$, centralization is required. Let $N = 4$ and $D_0 = 2$, the distance function $D(x,y)$ is given by

$$D(x,y) = \sqrt{\tilde{x}^2 + \tilde{y}^2},$$

where $\tilde{x} = (x + \frac{N}{2}) \mod N - \frac{N}{2} = \begin{cases} x & \text{if } 0 \le x < N/2 \\ N - x & \text{if } N/2 \le x < N \end{cases}$

and $\tilde{y} = (y + \frac{N}{2}) \mod N - \frac{N}{2} \begin{cases} y & \text{if } 0 \le y < N/2 \\ N - y & \text{if } N/2 \le y < N \end{cases}$. Hence

$$H_{LP} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

(b) The DFT of f is

$$\begin{aligned} F =& DFT(f) \\ =& \frac{1}{16} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \\ =& \frac{1}{8} \begin{pmatrix} 5 & -1+j & -1 & -1-j \\ -1+j & -1 & -1-j & 5 \\ -1 & -1-j & 5 & -1+j \\ -1-j & 5 & -1+j & -1 \end{pmatrix}. \end{aligned}$$

Therefore,

$$DFT(f_{sharp}) = (2 - H_{LP}) \odot F = \frac{1}{8} \begin{pmatrix} 5 & -1+j & -1 & -1-j \\ -1+j & -1 & -2-2j & 5 \\ -1 & -2-2j & 10 & -2+2j \\ -1-j & 5 & -2+2j & -1 \end{pmatrix}$$

And thus

$$\begin{aligned} f_{sharp} =& \frac{1}{8} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \begin{pmatrix} 5 & -1+j & -1 & -1-j \\ -1+j & -1 & -2-2j & 5 \\ -1 & -2-2j & 10 & -2+2j \\ -1-j & 5 & -2+2j & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \\ =& \frac{1}{8} \begin{pmatrix} 9 & -1 & 5 & -5 \\ -1 & 17 & -5 & 5 \\ 5 & -5 & 33 & -9 \\ -5 & 5 & -9 & 41 \end{pmatrix}. \end{aligned}$$

4. Apply the following filters:
the 3×3 mean filter;
the 3×3 median filter;
convolution filter $\frac{1}{12}\begin{pmatrix} 6 & 2 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ where the indices are starting from 0 , on the following periodically extended 4×4 images:

(a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & 3 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}$

Solution: Recall that the result g_{i} of the 3×3 mean filter applied on $f \in M_{M \times N}(\mathbb{R})$ is given by:

$$g_{\text{i}}(x, y) = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f(x+k, y+l);$$

the result g_{ii} of the 3×3 median filter applied on $f \in M_{M \times N}(\mathbb{R})$ is given by:

$$g_{\text{ii}}(x, y) = \text{median}\{f(x+k, y+l) : k, l \in \{-1, 0, 1\}\};$$

the result g_{iii} of the convolution filter applied on $f \in M_{4 \times 4}(\mathbb{R})$ is given by:

$$g_{\text{iii}}(x, y) = \frac{1}{2} f(x, y) + \frac{1}{6}[f(x+1, y) + f(x-1, y)] + \frac{1}{12}[f(x, y+1) + f(x, y-1)].$$

Hence the results from applying the filters on the given images are:

(a) i. $\frac{1}{9}\begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$

ii. $\mathbf{0}$

iii. $\frac{1}{4}\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}$

(b) i. $\frac{1}{9}\begin{pmatrix} 5 & 4 & 5 & 4 \\ 4 & 5 & 4 & 5 \\ 5 & 4 & 5 & 4 \\ 4 & 5 & 4 & 5 \end{pmatrix}$

ii. $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

iii. $\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

(c) i. $\frac{1}{9}\begin{pmatrix} -3 & 3 & 3 & 3 \\ 1 & 9 & 6 & 5 \\ 1 & 3 & 6 & 2 \\ 1 & 6 & 6 & 2 \end{pmatrix}$

ii. $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

iii. $\frac{1}{12}\begin{pmatrix} 0 & 7 & 16 & -2 \\ -1 & 1 & 14 & 2 \\ 4 & 19 & 14 & 3 \\ -9 & -1 & 6 & 0 \end{pmatrix}$

5. Consider an image denoising model to find $f : [a,b] \times [a,b] \to \mathbb{R}$ that minimizes:

$$E(f) = \int_a^b \int_a^b (f(x,y) - g(x,y))^2 dxdy + \int_a^b \int_a^b K(x,y)|\nabla f(x,y)|^2 dxdy$$

Assuming $f(x,y) = g(x,y) = 0$ for (x,y) on the boundary of $[a,b] \times [a,b]$. Suppose f minimizes $E(f)$. Show that f satisfies:

$$f(x,y) - g(x,y) - \nabla \cdot (K(x,y)\nabla f(x,y)) = 0 \text{ in } [a,b] \times [a,b]$$

Solution: Suppose f minimizes E. Let $D = [a,b] \times [a,b]$, then for any $\varphi : D \to \mathbb{R}$,

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\bigg|_{t=0} E(f + t\varphi) \\
&= 2\int_D \varphi(x,y)\,(f(x,y) - g(x,y)) + \int_D K(x,y)\frac{\partial}{\partial t}\bigg|_{t=0} \|\nabla(f + t\varphi)(x,y)\|^2 \\
&= 2\int_D \varphi(x,y)\,(f(x,y) - g(x,y)) \\
&\quad + \int_D K(x,y)\frac{\partial}{\partial t}\bigg|_{t=0} \left[\|\nabla f\|^2 + 2t\langle \nabla f(x,y), \nabla\varphi(x,y)\rangle + t^2\|\nabla\varphi(x,y)\|^2\right] \\
&= 2\int_D \{\varphi(x,y)\,(f(x,y) - g(x,y)) + K(x,y)\langle \nabla f(x,y), \nabla\varphi(x,y)\rangle\} \\
&= 2\int_D \varphi(x,y)\,(f(x,y) - g(x,y)) + 2\int_{\partial D} \varphi(x,y)K(x,y)\langle \nabla f(x,y), \vec{n}(x,y)\rangle \\
&\quad - 2\int_D \varphi(x,y)\nabla \cdot (K(x,y)\nabla f(x,y))\,.
\end{aligned}$$

Note that $f(x,y) = 0$ on ∂D, so the second term vanishes. Hence

$$0 = \int_D \varphi(x,y)\left[f(x,y) - g(x,y) - \nabla \cdot (K(x,y)\nabla f(x,y))\right].$$

Since the equation holds for any φ, for all $(x,y) \in D$ we have

$$f(x,y) - g(x,y) - \nabla \cdot (K(x,y)\nabla f(x,y)) = 0.$$