Math 3360: Mathematical Imaging

Assignment 4 Solution

1. Consider a $2N \times 2N$ image $I = (I(m,n))_{-N \leq m,n \leq N-1}$. The Butterworth high-pass filter H of squared radius D_0 and order n is applied on $DFT(I) = (\hat{I}(u,v))_{-N \leq u,v \leq N-1}$ to give $G(u,v)$. Suppose $\hat{I}(1,1) \neq 0$ and $\hat{I}(2,2) \neq 0$, and

$$G(1,1) = \frac{1}{37}\hat{I}(1,1) \text{ and } G(2,2) = \frac{1}{10}\hat{I}(2,2).$$

Find D_0 and n.

Solution: The given information implies

$$H(1,1) = \frac{1}{37} \text{ and } H(2,2) = \frac{1}{10}.$$

Hence

$$\begin{cases} 1 - \dfrac{D_0^n}{D_0^n + 2^n} &= \dfrac{1}{37}, \\ 1 - \dfrac{D_0^n}{D_0^n + 8^n} &= \dfrac{1}{10}, \end{cases} \text{ and thus } \begin{cases} 36 \cdot 2^n &= D_0^n, \\ 9 \cdot 8^n &= D_0^n. \end{cases}$$

Then $8^n = 4 \cdot 2^n$. Hence $n = 1$ and then $D_0 = 72$.

2. (a) Consider a $(2M+1) \times (2N+1)$ image $I = (I(m,n))_{0 \leq m \leq 2M,\, 0 \leq n \leq 2N}$, where $M, N > 200$. The Gaussian high-pass filter with standard deviation σ is applied to $DFT(I) = (\hat{I}(u,v))_{0 \leq m \leq 2M,\, 0 \leq n \leq 2N}$. Suppose $H(2,2) = \dfrac{1}{MN}$. Find σ^2.

   (b) Consider a Gaussian low-pass filter

$$H(u,v) = exp\left(-\frac{u^2+v^2}{2\sigma^2}\right).$$

   Suppose $H(4,2) = \frac{1}{\sqrt{e}}H(1,-3)$. Find σ^2.

Solution:

(a) The given information implies

$$1 - exp\left(-\frac{2^2+2^2}{2\sigma^2}\right) = \frac{1}{MN},$$

hence $\sigma^2 = \dfrac{4}{ln(MN) - ln(MN-1)}$.

(b) The given information implies

$$exp\left(-\frac{10}{\sigma^2}\right) = exp\left(-\frac{1}{2}\right) exp\left(-\frac{5}{\sigma^2}\right),$$

hence $\frac{10}{\sigma^2} = \frac{1}{2} + \frac{5}{\sigma^2}$ and so $\sigma^2 = 10$.

3. Suppose $g \in M_{N \times N}(\mathbb{R})$ is a blurred image capturing a static scene. Assume that g is given by:

$$g(u,v) = \frac{1}{\lambda}\sum_{k=0}^{\lambda-1} f(u-k,v) \text{ for } 0 \leq u,v \leq N-1,$$

where $\lambda \in \mathbb{N} \cap [1, N]$ and f is the underlying image (periodically extended). Show that $DFT(g)(u,v) = H(u,v)DFT(f)(u,v)$ for all $0 \le u, v \le N-1$, where $H(u,v)$ is the degradation function in the frequency domain given by:

$$H(u,v) = \begin{cases} \frac{1}{\lambda} \frac{\sin \frac{\lambda \pi u}{N}}{\sin \frac{\pi u}{N}} e^{-\pi j \frac{(\lambda-1)u}{N}} & \text{if } u \neq 0, \\ 1 & \text{if } u = 0. \end{cases}$$

Solution: Note that $g = h * f$, where

$$h(u,v) = \begin{cases} \frac{1}{\lambda} & \text{if } u \le \lambda - 1, v = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $0 \le u, v \le N-1$. Then

$$\begin{aligned} DFT(h)(u,v) &= \frac{1}{\lambda N^2} \sum_{k=0}^{\lambda-1} e^{-2\pi j \frac{ku}{N}} \\ &= \begin{cases} \frac{1}{\lambda N^2} \frac{1-e^{-2\pi j \frac{\lambda u}{N}}}{1-e^{-2\pi j \frac{u}{N}}} & \text{if } e^{-2\pi j \frac{u}{N}} \neq 1 \\ \frac{1}{N^2} & \text{if } e^{-2\pi j \frac{u}{N}} = 1 \end{cases} \\ &= \begin{cases} \frac{1}{\lambda N^2} \frac{e^{-\pi j \frac{\lambda u}{N}} (e^{\pi j \frac{\lambda u}{N}} - e^{-\pi j \frac{\lambda u}{N}})}{e^{-\pi j \frac{u}{N}} (e^{\pi j \frac{u}{N}} - e^{-\pi j \frac{u}{N}})} & \text{if } u \notin N\mathbb{Z} \\ \frac{1}{N^2} & \text{if } u \in N\mathbb{Z} \end{cases} \\ &= \begin{cases} \frac{1}{\lambda N^2} \frac{\sin \frac{\lambda \pi u}{N}}{\sin \frac{\pi u}{N}} e^{-\pi j \frac{(\lambda-1)u}{N}} & \text{if } u \neq 0 \\ \frac{1}{N^2} & \text{if } u \neq 0. \end{cases} \end{aligned}$$

Since $DFT(h * f) = N^2 DFT(h) \odot DFT(f)$,

$$H(u,v) = N^2 DFT(h)(u,v) = \begin{cases} \frac{1}{\lambda} \frac{\sin \frac{\lambda \pi u}{N}}{\sin \frac{\pi u}{N}} e^{-\pi j \frac{(\lambda-1)u}{N}} & \text{if } u \neq 0 \\ 1 & \text{if } u = 0 \end{cases}.$$

4. Consider a 4×4 periodically extended image $I = (I(k,l))_{0 \le k,l \le 3}$ given by:

$$I = \begin{pmatrix} a & a-2c & 0 & 0 \\ b-2c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix},$$

where $a, b, c \ge 0$.

Consider the modified direct filter T_1 with squared radius a and order b, which is defined by

$$T_1(u,v) = \frac{B(u,v)}{H_1(u,v) + \epsilon \cdot sgn(H_1(u,v))}$$

where $B(u,v) = \dfrac{1}{1 + (\frac{u^2+v^2}{a})^b}$, $\epsilon = 1$ and $H_1(u,v) = DFT(h_1)(u,v)$ with h_1 being a bluring convolution kernel $\begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Consider the constrained least square filter T_2 with parameter c, which is defined by

$$T_2(u,v) = \frac{1}{N^2} \frac{\overline{H_2(u,v)}}{|H_2(u,v)|^2 + c|P(u,v)|^2}$$

where $H_2 = DFT(h_2)$ with h_2 being the convolution kernel $\begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. and

$P = DFT(\tilde{h})$ with $\tilde{h}$ being the convolution kernel $\begin{pmatrix} -4 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Let $I_2(u,v) = T_2(u,v)DFT(I)(u,v)$.

Suppose $DFT(I)(0,0) = \frac{1}{2}$, $T_1(0,2) = \frac{8}{17}$, $DFT(I)(1,1) \neq 0$ and $I_2(1,1) = \frac{3j}{73}DFT(I)(1,1)$. Find a, b, c. (Here, $j = \sqrt{-1}$.)

Solution: First, from

$$DFT(I)(0,0) = \frac{1}{4^2}\sum_{k,l=0}^{3} I(k,l) = \frac{a + (a-2c) + (b-2c) + b}{4^2} = \frac{a+b-2c}{8} = \frac{1}{2},$$

we know that $a + b - 2c = 4$.

Secondly, recall that

$$H_1(u,v) = DFT(h_1)(u,v) = \frac{1}{4^2}\sum_{k,l=0}^{3} h_1(k,l)e^{-2\pi j\frac{k\cdot u + l\cdot v}{4}},$$

we can compute

$$H_1(0,2) = \frac{1}{4^2}\sum_{k,l=0}^{3} h_1(k,l)e^{-2\pi j\frac{2l}{4}} = \frac{1}{4^2}(\frac{1}{2} + \frac{1}{2}) = \frac{1}{16}$$

From

$$T_1(0,2) = \frac{B(0,2)}{H_1(0,2) + \epsilon \cdot sgn(H_1(0,2))} = \frac{1}{1 + (\frac{4}{a})^b} \Big/ \left(\frac{1}{16} + sgn(\frac{1}{16})\right) = \frac{16}{17}\frac{1}{1 + (\frac{4}{a})^b} = \frac{8}{17}$$

Therefore, $(\frac{4}{a})^b = 1$.

Besides,

$$\begin{aligned} H_2(1,1) &= DFT(h_2)(1,1) \\ &= \frac{1}{4^2}\sum_{k,l=0}^{3} h_2(k,l)e^{-2\pi j\frac{k+l}{4}} \\ &= \frac{1}{16}\cdot\frac{1}{3}(1 + e^{-\frac{1}{2}\pi j} + e^{-\pi j}) \\ &= -\frac{1}{48}j \end{aligned}$$

Similarly, $P(1,1) = DFT(\tilde{h})(1,1) = -\frac{1}{4}$.

Hence

$$T_2(1,1) = \frac{1}{4^2}\frac{\overline{H_2(1,1)}}{|H_2(1,1)|^2 + c|P(1,1)|^2} = \frac{1}{16}\frac{\frac{1}{48}j}{\frac{1}{48^2} + c\frac{1}{4^2}} = \frac{3j}{1+144c} = \frac{3j}{73}$$

From $\begin{cases} a + b - 2c = 4 \\ (\frac{4}{a})^b = 1 \\ 73 = 1 + 144c \end{cases}$ and $a, b, c \geq 0$,

we know that either $a = 4, b = 1, c = \frac{1}{2}$ or $a = 5, b = 0, c = \frac{1}{2}$

5. The constrained least square filtering aims to find a vectorized image $\vec{f}$ of a $N \times N$ image f that minimizes: $E(\vec{f}) = (L\vec{f})^T(L\vec{f})$ subject to the constraint: $[\vec{g} - H\vec{f}]^T[\vec{g} - H\vec{f}] = \epsilon$, for some block-circulant matrices H and L. ϵ is a fixed parameter greater than 0.

 (a) Let $W = W_2 \otimes W_2$, where $W_2(k, n) = \frac{1}{\sqrt{2}} e^{\pi jkn}$ for $0 \leq k, n \leq 1$ and $\otimes$ is the Kronecker product. Given that $H = W\Lambda_H W^{-1}$ and $L = W\Lambda_L W^{-1}$, where

$$\Lambda_H = \begin{pmatrix} h_0 & 0 & 0 & 0 \\ 0 & h_1 & 0 & 0 \\ 0 & 0 & h_2 & 0 \\ 0 & 0 & 0 & h_3 \end{pmatrix} \text{ and } \Lambda_L = \begin{pmatrix} l_0 & 0 & 0 & 0 \\ 0 & l_1 & 0 & 0 \\ 0 & 0 & l_2 & 0 \\ 0 & 0 & 0 & l_3 \end{pmatrix}.$$

 where $h_i, l_i \in \mathbb{R}^+, 0 \leq i \leq 3$

 Let $\vec{g} = \mathcal{S}(g)$, where g is a 2×2 image and $\mathcal{S}$ is the stacking operator.

  i. Show that H is block-circulant

  ii. Show that $W^{-1}\mathcal{S}(h) = 2\mathcal{S}(\hat{h})$ for any 2×2 image h.

 (b) Show that the optimal solution $\vec{f} = \mathcal{S}(f)$ that solves the constrained least square problem satisfies $[\lambda H^T H + L^T L]\vec{f} = \lambda H^T \vec{g}$ for some parameter λ. Hence, find $DFT(f)$ in term of $DFT(g)$, $h_i, l_i, 0 \leq i \leq 3$ and λ. You may assume $\lambda > 0$. Please show your answer with details.

Solution:

 (a) Here we prove the general case for any $N \in \mathbb{N}^+$.

  i. Consider W, W^{-1}, H and Λ_H as $N \times N$ block matrices and every block of them is a $N \times N$ matrix. We have

$$\begin{aligned} W &= W_N \otimes W_N = \frac{1}{\sqrt{N}} \left(e^{\frac{2\pi j}{N}kl} W_N\right)_{0 \leq k,l \leq N-1}, \\ W^{-1} &= W_N^{-1} \otimes W_N^{-1} = \frac{1}{\sqrt{N}} \left(e^{-\frac{2\pi j}{N}kl} W_N^{-1}\right)_{0 \leq k,l \leq N-1}, \\ H &= (H_{k,l})_{0 \leq k,l \leq N-1}, \\ \Lambda_H &= \begin{pmatrix} \Lambda_0 & 0 & \cdots & 0 \\ 0 & \Lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda_{N-1} \end{pmatrix}, \end{aligned}$$

  where

$$\Lambda_i = \begin{pmatrix} h_{iN} & 0 & \cdots & 0 \\ 0 & h_{iN+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_{(i+1)N-1} \end{pmatrix}$$

  for any $i = 0, 1, \cdots, N-1$. Then compute

$$\begin{aligned} H &= W\Lambda_H W^{-1} \\ &= \frac{1}{N} \left(e^{\frac{2\pi j}{N}kl} W_N\right)_{0 \leq k,l \leq N-1} \Lambda_H \left(e^{-\frac{2\pi j}{N}lk} W_N\right)_{0 \leq k,l \leq N-1} \\ &= \frac{1}{N} \left(e^{\frac{2\pi j}{N}kl} W_N \Lambda_l\right)_{0 \leq k,l \leq N-1} \left(e^{-\frac{2\pi j}{N}lk} W_N\right)_{0 \leq k,l \leq N-1} \\ &= \frac{1}{N} \left(\sum_{m=0}^{N-1} e^{\frac{2\pi j}{N}km} e^{-\frac{2\pi j}{N}ml} W_N \Lambda_m W_N^{-1}\right)_{0 \leq k,l \leq N-1} \\ &= \frac{1}{N} \left(W_N \left(\sum_{m=0}^{N-1} e^{\frac{2\pi j}{N}m(k-l)} \Lambda_m\right) W_N^{-1}\right)_{0 \leq k,l \leq N-1}. \end{aligned}$$

  So $H_{k,l} = \frac{1}{N} W_N \left(\sum_{m=0}^{N-1} e^{\frac{2\pi j}{N}m(k-l)} \Lambda_m\right) W_N^{-1}$ and $H_{k+1,l+1} = H_{k,l}$, which means H is block-circulant.

ii. We know that $\hat{h} = UhU$, where $U = (u_{kl})_{0\le k,l\le N-1}$ and $u_{kl} = \frac{1}{N}e^{-\frac{2\pi j}{N}kl}$. It's easy to find $u_{kl} = \frac{1}{\sqrt{N}}W_N^{-1}(k,l)$, so $W_N^{-1} = \sqrt{N}U$. By the property of sperable image transformation, we have

$$\begin{aligned} W^{-1}\mathcal{S}(h) &= W_N^{-1} \otimes W_N^{-1}\mathcal{S}(h) \\ &= \mathcal{S}(W_N^{-1}h(W_N^{-1})^T) \\ &= \mathcal{S}(W_N^{-1}hW_N^{-1}) \\ &= \mathcal{S}(NUhU) \\ &= N\mathcal{S}(\hat{h}) \end{aligned}$$

(b) We know the minimizer must satisfy

$$\mathcal{D} = \frac{\partial}{\partial \vec{f}}[\vec{f}^T L^T L\vec{f} + \lambda(\vec{g} - H\vec{f})^T(\vec{g} - H\vec{f})] = 0$$

where λ is the Lagrange multiplier. Easy to check

$$\begin{aligned} &\mathcal{D} = 0 \\ &\Rightarrow (2L^T L)\vec{f} + \lambda(-H^T\vec{g} - H^T\vec{g} + 2H^T H\vec{f}) = 0 \\ &\Rightarrow (\lambda H^T H + L^T L)\vec{f} = \lambda H^T\vec{g}. \end{aligned}$$

Since L is also block-circulant, it is also diagonalizable by W. Then

$$\begin{aligned} \lambda H^T H + L^T L &= \lambda H^* H + L^* L \\ &= \lambda(W\Lambda_H W^{-1})^*(W\Lambda_H W^{-1}) + (W\Lambda_L W^{-1})^*(W\Lambda_L W^{-1}) \\ &= \lambda W\Lambda_H^*\Lambda_H W^{-1} + W\Lambda_L^*\Lambda_L W^{-1} \end{aligned}$$

and $\lambda H^T = \lambda H^* = \lambda(W\Lambda_H W^{-1})^* = \lambda W\Lambda_H^* W^{-1}$. Hence

$$\begin{aligned} (\lambda W\Lambda_H^*\Lambda_H W^{-1} + W\Lambda_L^*\Lambda_L W^{-1})\mathcal{S}(f) &= \lambda W\Lambda_H^* W^{-1}\mathcal{S}(g), \\ (\lambda\Lambda_H^*\Lambda_H + \Lambda_L^*\Lambda_L)\mathcal{S}(DFT(f)) &= \lambda\Lambda_H^*\mathcal{S}(DFT(g)) \end{aligned}$$

and thus

$$DFT(f)(u,v) = \frac{\lambda h_{u+2v}}{\lambda h_{u+2v}^2 + l_{u+2v}^2} DFT(g)(u,v).$$