Math 3360: Mathematical Imaging

Assignment 3 Solutions

1. Recall that the discrete Fourier transformation(DFT) $\hat{g}$ of an $N \times N$ image g is defined as

$$\hat{g}(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l) e^{-2\pi\sqrt{-1}\frac{km+ln}{N}}.$$

   (a) Write down the Fourier transformation matrix U for a 4×4 image, i.e. the matrix such that the discrete Fourier transformation of f is UfU.

   (b) Compute the DFT of the following 4×4 image

$$g = (g(k,l))_{0 \le k,l \le 3} = \begin{pmatrix} 4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (c) Let $f \in M_{4\times 4}(\mathbb{R})$ such that $\widehat{f * g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, compute f.

   Solution:

   (a) The DFT matrix for $n = 4$ is

$$U = (\frac{e^{-2\pi i \frac{kl}{4}}}{4})_{kl} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -j \end{pmatrix}.$$

   (b)

$$\hat{g} = UgU = \frac{1}{16} \begin{pmatrix} 9 & 4+i & 7 & 4-i \\ 4+i & -1 & 4-i & 1 \\ 7 & 4-i & 9 & 4+i \\ 4-i & 1 & 4+i & -1 \end{pmatrix}.$$

   (c) We have $\widehat{f * g} = 16\hat{f} \odot \hat{g}$, it's easy to find $\hat{f} = \frac{1}{9} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, then we do iDFT and get

$$f = (4U^*)\hat{f}(4U^*) = \frac{1}{9} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

2. Let $g = (g(k,l))_{0 \le k,l \le N}$ be an $N \times N$ image, and denote its reflection about the line $l = -\frac{1}{2}$ by $\tilde{g} = (\tilde{g}(k,l))_{0 \le k \le N-1, 1-N \le l \le 0}$. That is,

$$\tilde{g}(k,l) = g(k, -1-l) \text{ for } 0 \le k \le N-1 \text{ and } -N \le l \le -1.$$

Prove that

$$DFT(\tilde{g})(m,n) = e^{2\pi j \frac{n}{N}} \hat{g}(m,-n).$$

Solution: For the LHS, we have

$$\begin{aligned}
&DFT(\tilde{g})(m,n)\\
=&\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=-N}^{-1}\tilde{g}(k,l)e^{-2\pi j\frac{km+ln}{N}}\\
=&\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=-N}^{-1}g(k,-1-l)e^{-2\pi j\frac{km+ln}{N}}\\
=&\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}g(k,l)e^{-2\pi j\frac{km-ln-n}{N}}.
\end{aligned}$$

And the RHS is

$$\begin{aligned}
&e^{2\pi j\frac{n}{N}}\hat{g}(m,-n)\\
=&e^{2\pi j\frac{n}{N}}\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}g(k,l)e^{-2\pi j\frac{km-ln}{N}}\\
=&\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}g(k,l)e^{-2\pi j\frac{km-ln-n}{N}}.
\end{aligned}$$

Since LHS = RHS, the original equation is proved.

3. Let $\hat{f}$ be the discrete Fourier transformation of $M \times N$ image f. Prove that $\hat{f} * \hat{g} = \widehat{f \odot g}$, where $f \odot g(k,l) = f(k,l)g(k,l)$.

Solution: We have that

$$\begin{aligned}
\hat{f}(m,n) &= \frac{1}{MN}\sum_{k=1}^{M}\sum_{l=1}^{N}f(k,l)e^{-2\pi j(\frac{km}{M}+\frac{ln}{N})},\\
\hat{g}(m,n) &= \frac{1}{MN}\sum_{k=1}^{M}\sum_{l=1}^{N}g(k,l)e^{-2\pi j(\frac{km}{M}+\frac{ln}{N})},\\
\widehat{f \odot g}(m,n) &= \frac{1}{MN}\sum_{k=1}^{M}\sum_{l=1}^{N}f(k,l)g(k,l)e^{-2\pi j(\frac{km}{M}+\frac{ln}{N})}.
\end{aligned}$$

On the other side

$$\begin{aligned}
&\hat{f} * \hat{g}(m,n)\\
=&\sum_{k=1}^{M}\sum_{l=1}^{N}\hat{f}(k,l)\hat{g}(m-k,n-l)\\
=&\sum_{k=1}^{M}\sum_{l=1}^{N}(\frac{1}{MN}\sum_{x=1}^{M}\sum_{y=1}^{N}f(x,y)e^{-2\pi j(\frac{xk}{M}+\frac{yl}{N})})(\frac{1}{MN}\sum_{x'=1}^{M}\sum_{y'=1}^{N}g(x',y')e^{-2\pi j(\frac{x'(m-k)}{M}+\frac{y'(n-l)}{N})})\\
=&\frac{1}{M^2N^2}\sum_{k,x,x'=1}^{M}\sum_{l,y,y'=1}^{N}f(x,y)g(x',y')e^{-2\pi j(\frac{xk+x'(m-k)}{M}+\frac{yl+y'(n-l)}{N})}\\
=&\frac{1}{M^2N^2}\sum_{x,x'=1}^{M}\sum_{y,y'=1}^{N}f(x,y)g(x',y')e^{-2\pi j(\frac{x'm}{M}+\frac{y'n}{N})}\sum_{k=1}^{M}e^{-2\pi j\frac{(x-x')k}{M}}\sum_{l=1}^{N}e^{-2\pi j\frac{(y-y')l}{N}}\\
=&\frac{1}{M^2N^2}\sum_{x,x'=1}^{M}\sum_{y,y'=1}^{N}f(x,y)g(x',y')e^{-2\pi j(\frac{x'm}{M}+\frac{y'n}{N})}M\delta(x-x')N\delta(y-y')\\
=&\frac{1}{MN}\sum_{x=1}^{M}\sum_{y=1}^{N}f(x,y)g(x,y)e^{-2\pi j(\frac{xm}{M}+\frac{yn}{N})}.
\end{aligned}$$

Therefore, $\hat{f} * \hat{g} = \widehat{f \odot g}(m, n)$.

4. The discrete Laplace operator Δ on a periodically extended $N \times N$ image $N \geq 3$ can be writen as:

$$\Delta f(x, y) = 2f(x+1, y) + 2f(x-1, y) + f(x, y+1) + f(x, y-1) - 6f(x, y).$$

Prove that $DFT(\Delta f)(u, v) = H(u, v)F(u, v)$ for some $H \in M_{N\times N}(\mathbb{C})$, where $F = DFT(f)$. Find $H(u, v)$ as a trigonometric polynomial in $\frac{\pi u}{N}$ and $\frac{\pi v}{N}$, i.e. as a polynomial in $\sin\frac{\pi u}{N}$, $\sin\frac{\pi v}{N}$, $\cos\frac{\pi u}{N}$ and $\cos\frac{\pi v}{N}$.

Solution: Note that $\Delta f = h * f$ where

$$h(x, y) = \begin{cases} -6, (x, y) = (N, N) \\ 2, (x, y) = (N-1, N) \text{ or } (1, N) \\ 1, (x, y) = (N, N-1) \text{ or } (N, 1) \\ 0, \text{otherwise} \end{cases}.$$

Then $DFT(\Delta f) = DFT(h * f) = N^2 DFT(h) \odot DFT(f)$, let $H = N^2 DFT(h)$, we have $DFT(\Delta f)(u, v) = H(u, v)F(u, v)$. And

$$\begin{aligned} H(u, v) &= \sum_{x=1}^{N}\sum_{y=1}^{N} h(x, y)e^{-2\pi j\frac{xu+yv}{N}} \\ &= -6e^{-2\pi j\frac{Nu+Nv}{N}} + 2e^{-2\pi j\frac{(N-1)u+Nv}{N}} + 2e^{-2\pi j\frac{u+Nv}{N}} + e^{-2\pi j\frac{Nu+(N-1)v}{N}} + e^{-2\pi j\frac{Nu+v}{N}} \\ &= -6 + 2e^{2\pi j\frac{u}{N}} + 2e^{-2\pi j\frac{u}{N}} + e^{2\pi j\frac{v}{N}} + e^{-2\pi j\frac{v}{N}} \\ &= -6 + 2e^{2\pi j\frac{u}{N}} + 2e^{-2\pi j\frac{u}{N}} + e^{2\pi j\frac{v}{N}} + e^{-2\pi j\frac{v}{N}} \\ &= -6 + 4\cos 2\frac{\pi u}{N} + 2\cos 2\frac{\pi v}{N} \\ &= -6 + 4\cos^2\frac{\pi u}{N} - 4\sin^2\frac{\pi u}{N} + 2\cos^2\frac{\pi v}{N} - 2\sin^2\frac{\pi v}{N}. \end{aligned}$$

5. **(Optimal) Programming exercise:** Please read the MATLAB file **or** the Python file in the attached zip file carefully. There are missing lines in the file. You can either choose MATLAB or Python to finish. Add the missing lines by yourself and test the file using the given image. (Note: In this coding assignment, we discuss the image processing of grayscale images only.)

Coding instruction:

Recall that DFT can be rewritten as matrix multiplication.

$$\hat{g} = UgU$$

where $U_{\alpha\beta} = \dfrac{1}{N}e^{-2\pi j\frac{\alpha\beta}{N}}$ where $0 \leq \alpha, \beta \leq N-1$, and $U = (U_{\alpha\beta})_{0\leq\alpha,\beta\leq N-1} \in M_{N\times N}(\mathbb{C})$. In this coding assignment, you are required to **reconstruct the image** from the given modified Fourier coefficients $\hat{g}$. You **are not allowed** to use the built-in MATLAB function *ifft2* or any Python Fourier transform module such as *numpy.fft* module and *scipy.fft*. Please submit your code as well as the reconstructed image.

Solution: The codes(Python and MATLAB versions) have been upload to course website.