Math 3360: Mathematical Imaging

Assignment 2 solutions

1. (a) Let $A, B \in M_{4\times 4}(\mathbb{R})$ and the image transformation $\mathcal{O} : M_{4\times 4}(\mathbb{R}) \to M_{4\times 4}(\mathbb{R})$ is defined by:

$$\mathcal{O}(f) = AfB,$$

please show that the transformation matrix H of $\mathcal{O}$ is given by:

$$H = B^T \otimes A.$$

(b) **(Optimal)** In more general cases, let $A, B \in M_{n\times n}(\mathbb{R})$ and the image transformation $\mathcal{O} : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ is defined by:

$$\mathcal{O}(f) = AfB,$$

please show that the transformation matrix H of $\mathcal{O}$ is also given by:

$$H = B^T \otimes A.$$

Solution: Here, we only need to prove the general situation (b), then (a) is just a special case for $n = 4$. Let $A = (a_{ij})$, $B = (b_{ij})$ and $g = \mathcal{O}(f) \in M_{n\times n}(\mathbb{R})$, then we have

$$g_{\alpha,\beta} = \sum_{x=1}^{n} a_{\alpha x}(\sum_{y=1}^{n} f(x,y)b_{y\beta}) = \sum_{x=1}^{n}\sum_{y=1}^{n} a_{\alpha x}b_{y\beta}f(x,y),$$

Which means $h^{\alpha,\beta}(x,y) = a_{\alpha x}b_{y\beta}$. Since the transformation matrix

$$H = \begin{pmatrix} \begin{pmatrix} & x\to \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=1 \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x\to \\ \alpha\downarrow & \begin{pmatrix} y=n \\ \beta=1 \end{pmatrix} \end{pmatrix} \\ \vdots & \ddots & \vdots \\ \begin{pmatrix} & x\to \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=n \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x\to \\ \alpha\downarrow & \begin{pmatrix} y=n \\ \beta=n \end{pmatrix} \end{pmatrix} \end{pmatrix}$$

and

$$\begin{pmatrix} & x\to \\ \alpha\downarrow & \begin{pmatrix} y=i \\ \beta=j \end{pmatrix} \end{pmatrix} = \begin{pmatrix} h^{1,j}(1,i) & \cdots & h^{1,j}(n,i) \\ \vdots & \ddots & \vdots \\ h^{n,j}(1,i) & \cdots & h^{n,j}(n,i) \end{pmatrix} = \begin{pmatrix} a_{11}b_{ij} & \cdots & a_{1n}b_{ij} \\ \vdots & \ddots & \vdots \\ a_{n1}b_{ij} & \cdots & a_{nn}b_{ij} \end{pmatrix} = b_{ij}A.$$

Therefore

$$H = \begin{pmatrix} b_{11}A & \cdots & b_{n1}A \\ \vdots & \ddots & \vdots \\ b_{1n}A & \cdots & b_{nn}A \end{pmatrix} = B^T \otimes A.$$

2. Compute the singular value decomposition(SVD) of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Please show all your steps in detail.

Solution: We first compute the characteristic polynomial of A^TA

$$A^TA = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

the characteristic polynomial of A^TA is given by

$$\det(A^TA - \lambda i) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - 2).$$

So the eigenvalues of A^TA is $\lambda_1 = 2$ and $\lambda_2 = 1$. The corresponding eigenvectors are $\vec{u}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then

$$\vec{v}_1 = \frac{A^T\vec{u}_1}{\sqrt{\lambda_1}} = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

$$\vec{v}_2 = \frac{A^T\vec{u}_2}{\sqrt{\lambda_2}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\vec{v}_3 = \frac{\vec{v}_1 \times \vec{v}_2}{\|\vec{v}_1 \times \vec{v}_2\|} = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Therefore, $A = U\Sigma V^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$.

3. Let $H_n(t)$ be the n-th Haar function, where $n \in \mathbb{N} \cup \{0\}$.

   (a) Give the definition of $H_n(t)$.

   (b) Write down the Haar transformation matrix $\tilde{H}$ for 4×4 images.

   (c) Suppose $A = \begin{pmatrix} 5 & 4 & 6 & 6 \\ 6 & 1 & 6 & 3 \\ 1 & 2 & 1 & 5 \\ 6 & 4 & 6 & 1 \end{pmatrix}$. Compute the Haar transform A_{Haar} of A, and compute the reconstructed image $\tilde{A}$ after setting the two smallest (in absolute value) nonzero entries of A_{Haar} to 0.

Solution:

(a) Haar function is defined as

$$H_0(t) = \begin{cases} 1, 0 \le t < 1 \\ 0, \text{otherwise} \end{cases},$$

$$H_1(t) = \begin{cases} 1, 0 \le t < \frac{1}{2} \\ -1, \frac{1}{2} \le t < 1 \\ 0, \text{otherwise} \end{cases},$$

$$H_n(t) = H_{2^p+n_0}(t) = \begin{cases} \sqrt{2}^p, \frac{n_0}{2^p} \le t < \frac{n_0+0.5}{2^p} \\ -\sqrt{2}^p, \frac{n_0+0.5}{2^p} \le t < \frac{n_0+1}{2^p} \\ 0, \text{otherwise} \end{cases}$$

where $p = 1, 2, \cdots$ and $n_0 = 0, 1, 2, \cdots, 2^p - 1$.

(b)

$$\tilde{H} = \frac{1}{\sqrt{4}}\begin{pmatrix} H_1(\frac{0}{4}) & H_1(\frac{1}{4}) & H_1(\frac{2}{4}) & H_1(\frac{3}{4}) \\ H_2(\frac{0}{4}) & H_2(\frac{1}{4}) & H_2(\frac{2}{4}) & H_2(\frac{3}{4}) \\ H_3(\frac{0}{4}) & H_3(\frac{1}{4}) & H_3(\frac{2}{4}) & H_3(\frac{3}{4}) \\ H_4(\frac{0}{4}) & H_4(\frac{1}{4}) & H_4(\frac{2}{4}) & H_4(\frac{3}{4}) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}.$$

(c) We have

$$A_{\text{Haar}} = \tilde{H} A \tilde{H}^T = \frac{1}{4} \begin{pmatrix} 63 & -5 & 7\sqrt{2} & 4\sqrt{2} \\ 11 & -5 & 5\sqrt{2} & 2\sqrt{2} \\ 5\sqrt{2} & -\sqrt{2} & -8 & -6 \\ -8\sqrt{2} & -6\sqrt{2} & -6 & -18 \end{pmatrix}.$$

Then we set the two smallest entries to 0 and get

$$A'_{\text{Haar}} = \frac{1}{4} \begin{pmatrix} 63 & -5 & 7\sqrt{2} & 4\sqrt{2} \\ 11 & -5 & 5\sqrt{2} & 0 \\ 5\sqrt{2} & 0 & -8 & -6 \\ -8\sqrt{2} & -6\sqrt{2} & -6 & -18 \end{pmatrix},$$

and the reconstructed image is

$$\tilde{A} = \tilde{H}^T A'_{\text{Haar}} \tilde{H} = \frac{1}{16} \begin{pmatrix} 82 & 66 & 90 & 98 \\ 94 & 14 & 94 & 54 \\ 16 & 32 & 20 & 76 \\ 96 & 64 & 100 & 12 \end{pmatrix}.$$

4. For an $n \times n$ image g of real entries, let $g = UfV^T$, where $U, V, f \in M_{n\times n}(\mathbb{R})$.

(a) Show that

$$g = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} \vec{u_i} \vec{v_j}^T,$$

where

$$U = \begin{pmatrix} \vec{u_1} & \vec{u_2} & \cdots & \vec{u_n} \end{pmatrix}, V^T = \begin{pmatrix} \vec{v_1}^T \\ \vec{v_2}^T \\ \vdots \\ \vec{v_n}^T \end{pmatrix},$$

(b) Show that if f is diagonal, then the trace of g is given by

$$\text{tr}(g) = \sum_{k=1}^{n} g_{kk} = \sum_{k=1}^{n} \sum_{l=1}^{n} f_{ll} u_{kl} v_{kl}.$$

Solution:

(a) Let $f = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} E_{ij}$, where $E_{ij} \in M_{n\times n}(\mathbb{R})$ is 1 only in i,j-entry and is 0 elsewhere, then

$$\begin{aligned} g = UfV^T &= \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} U E_{ij} V^T \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} \begin{pmatrix} 0 & \cdots & \vec{u_i} & \cdots & 0 \end{pmatrix} \begin{pmatrix} \vec{v_1}^T \\ \vec{v_2}^T \\ \vdots \\ \vec{v_n}^T \end{pmatrix} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} \vec{u_i} \vec{v_j}^T. \end{aligned}$$

(b) Since f is diagonal, $f_{ij} = 0$ when $i \neq j$ and

$$g = \sum_{l=1}^{n} f_{ll} \vec{u_l} \vec{v_l}^T.$$

Therefore

$$g_{kk} = \sum_{l=1}^{n} f_{ll} u_{kl} v_{kl}$$

and

$$tr(g) = \sum_{k=1}^{n} = \sum_{k=1}^{n}\sum_{l=1}^{n} f_{ll}u_{kl}v_{kl}$$

5. Suppose $J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, Compute $\hat{J} = DFT(J)$.

   Solution: The DFT matrix for $n = 4$ is

$$U = (\frac{e^{-2\pi i \frac{kl}{4}}}{4})_{kl} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -j \end{pmatrix}.$$

   Therefore

$$\hat{J} = UJU = \frac{1}{16}\begin{pmatrix} 6 & -1-i & 0 & -1+i \\ -1-i & 2i & 1-i & 4 \\ 0 & 1-i & 2 & 1+i \\ -1+i & 4 & 1+i & -2i \end{pmatrix}.$$

6. **(Optimal) Programming exercise:** Compress a digital image using SVD, please try to show the rank-k approximations with $k = 5, 10, 50$ respectively.

   Hint: You can use any programming language (python, matlab, R and so on) with any third-party library, you DON'T need to implement the SVD algorithm yourself. Please submit the following as your solutions:

   1. your code,
   2. original image,
   3. rank-k approximations for $k = 5, 10, 50$.

   Solution: The codes(Python and MATLAB versions) have been upload to course website.