Math 3360: Mathematical Imaging

Assignment 1 solutions

1. Prove or disprove if the following image transformation $\mathcal{O} : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ is linear.

   (a) Let $a \in \mathbb{R}$, $A \in M_{N\times N}(\mathbf{R})$. For any $f \in M_{N\times N}(\mathbf{R})$, $\mathcal{O}(f) = af + Af^T$, where f^T is the transpose of f.

   (b) Let $A \in M_{N\times N}(\mathbf{R})$. For any $f \in M_{N\times N}(\mathbf{R})$, $\mathcal{O}(f) = fAf$.

   (c) Let $k \in M_{N\times N}(\mathbf{R})$. For any $f \in M_{N\times N}(\mathbf{R})$, $\mathcal{O}(f) = k * f$, where $*$ denote the discrete convolution.

   Solution:

   (a) For all $f, g \in M_{N\times N}(\mathbf{R})$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned}\mathcal{O}(\alpha f + \beta g) =& a(\alpha f + \beta g) + A(\alpha f + \beta g)^T \\ =& \alpha(af + Af^T) + \beta(ag + Ag^T) \\ =& \alpha\mathcal{O}(f) + \beta\mathcal{O}(g).\end{aligned}$$

   Thus $\mathcal{O}$ is linear.

   (b) For all $f, g \in M_{N\times N}(\mathbf{R})$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned}\mathcal{O}(\alpha f + \beta g) =& (\alpha f + \beta g)A(\alpha f + \beta g) \\ =& (\alpha f + \beta g)(\alpha Af + \beta Ag) \\ =& \alpha f(\alpha Af + \beta Ag) + \beta g(\alpha Af + \beta Ag) \\ =& \alpha^2 fAf + \alpha\beta fAg + \alpha\beta gAf + \beta^2 gAg,\end{aligned}$$

   but

$$\alpha\mathcal{O}(f) + \beta\mathcal{O}(g) = \alpha fAf + \beta gAg.$$

   Thus $\mathcal{O}$ is not linear.

   (c) For all $f, g \in M_{N\times N}(\mathbf{R})$ and $\alpha, \beta \in \mathbb{R}$, we have for all $1 \le m, n \le N$

$$\begin{aligned}\mathcal{O}(\alpha f + \beta g)(m, n) =& k * (\alpha f + \beta g)(m, n) \\ =& \sum_{x=1}^{N}\sum_{y=1}^{N} k(x, y)(\alpha f + \beta g)(m - x, n - y) \\ =& \alpha\sum_{x=1}^{N}\sum_{y=1}^{N} k(x, y)f(m - x, n - y) \\ & + \beta\sum_{x=1}^{N}\sum_{y=1}^{N} k(x, y)g(m - x, n - y) \\ =& \alpha k * f(m, n) + \beta k * g(m, n) \\ =& \alpha\mathcal{O}(f)(m, n) + \beta\mathcal{O}(g)(m, n)\end{aligned}$$

   Thus $\mathcal{O}$ is linear.

2. Let $A = (a_{ij})_{1\le i,j\le 2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = (b_{ij})_{1\le i,j\le 2} = \begin{pmatrix} 1 & 3 \\ 5 & 6 \end{pmatrix}$. Define the image transformation $\mathcal{O} = M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by $\mathcal{O}(f) = AfB$. Let

$$H^{1,2} = \begin{pmatrix} h^{1,2}(1,1) & h^{1,2}(1,2) \\ h^{1,2}(2,1) & h^{1,2}(2,2) \end{pmatrix},$$

where $h^{\alpha,\beta}(x,y)$ is the point spread function of $\mathcal{O}$. Compute $H^{1,2}$.

Solution: By the definition, we have

$$\mathcal{O}(f)(1,2) = (fB)_{12} + 2(fB)_{22} = 3f_{11} + 6f_{12} + 6f_{21} + 12f_{22},$$

so

$$H^{1,2} = \begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix}$$

3. Let $f = (f_{ij})_{1\le i,j\le 3} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = (b_{ij})_{1\le i,j\le 3} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & 4 \end{pmatrix}$.

   (a) Compute $f * B$, where $*$ denote the discrete convolution.

   (b) Let $g = f * B \in M_{3\times 3}(\mathbb{R})$, show that for all $1 \le \alpha, \beta \le 3$

   $$g(\alpha,\beta) = 4f_{\alpha,\beta} - f_{\alpha+1,\beta} - f_{\alpha-1,\beta} - f_{\alpha,\beta+1} - f_{\alpha,\beta-1},$$

   where $g(\alpha,\beta)$ are the α-th row, β-th column of g.

   Solution:

   (a) We have

   $$\begin{aligned}
   f * B(1,1) &= f_{11}b_{33} + f_{12}b_{32} + f_{13}b_{31} = 4f_{11} - f_{12} - f_{13} = 2 \\
   f * B(1,2) &= f_{11}b_{31} + f_{12}b_{33} + f_{13}b_{32} + f_{32}b_{13} = 4f_{12} - f_{11} - f_{13} - f_{32} = 1 \\
   f * B(1,3) &= f_{11}b_{32} + f_{12}b_{31} + f_{13}b_{33} = 4f_{13} - f_{11} - f_{13} = 2 \\
   f * B(2,1) &= f_{11}b_{13} = -f_{11} = -1 \\
   f * B(2,2) &= f_{12}b_{13} + f_{32}b_{23} = -f_{12} - f_{32} = -2 \\
   f * B(2,3) &= f_{13}b_{13} = -f_{13} = -1 \\
   f * B(3,1) &= f_{11}b_{23} + f_{32}b_{32} = -f_{11} - f_{32} = -2 \\
   f * B(3,2) &= f_{12}b_{23} + f_{32}b_{33} = 4f_{32} - f_{12} = 3 \\
   f * B(3,3) &= f_{13}b_{23} + f_{32}b_{31} = -f_{13} - f_{32} = -2,
   \end{aligned}$$

   thus $f * B = \begin{pmatrix} 2 & 1 & 2 \\ -1 & -2 & -1 \\ -2 & 3 & -2 \end{pmatrix}$.

   (b) This have been directly checked in (a), here we show another method.

   $$\begin{aligned}
   g(\alpha,\beta) =& \sum_{x=1}^{3}\sum_{y=1}^{3} f_{x,y} b_{\alpha-x,\beta-y} = \sum_{x=1}^{3}\sum_{y=1}^{3} b_{x,y} f_{\alpha-x,\beta-y} \\
   =& 4f_{\alpha,\beta} - f_{\alpha+1,\beta} - f_{\alpha-1,\beta} - f_{\alpha,\beta+1} - f_{\alpha,\beta-1}.
   \end{aligned}$$

4. Define a linear image transformation $\mathcal{O} : M_{N\times N}(\mathbf{R}) \to M_{N\times N}(\mathbf{R})$ by

   $$\mathcal{O}(f)(\alpha,\beta) = \frac{f(\alpha+1,\beta) + 2f(\alpha-1,\beta) + 3f(\alpha,\beta+1) + f(\alpha,\beta-1) - 8f(\alpha,\beta)}{4}.$$

   Show that $\mathcal{O}(f) = k * f$ for some $k \in M_{N\times N}(\mathbf{R})$ and find this k.

Solution: Let $k(x,y)$ be the x-th row, y-th column of k,

$$\begin{aligned}
\mathcal{O}(f)(\alpha,\beta) =& \sum_{x=1}^{N}\sum_{y=1}^{N} k(x,y) f(\alpha - x, \beta - y) \\
=& \frac{f(\alpha+1,\beta) + 2f(\alpha-1,\beta) + 3f(\alpha,\beta+1) + f(\alpha,\beta-1) - 8f(\alpha,\beta)}{4} \\
=& \frac{1}{4} f(\alpha - N + 1, \beta - N) + \frac{1}{2} f(\alpha - 1, \beta - N) + \frac{3}{4} f(\alpha - N, \beta - N + 1) \\
&+ \frac{1}{4} f(\alpha - N, \beta - 1) - 2f(\alpha - N, \beta - N),
\end{aligned}$$

then we have $k(N-1,N) = \frac{1}{4}$, $k(1,N) = \frac{1}{2}$, $k(N,N-1) = \frac{3}{4}$, $k(N,1) = \frac{1}{4}$, $k(N,N) = -2$ and the rest $k(x,y) = 0$, which means

$$k = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \cdots & 0 & \frac{3}{4} & -2 \end{pmatrix}.$$

5. Compute the singular value decomposition(SVD) of

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Please show all your steps in detail.

Solution: we first compute the characteristic polynomial of $A^T A$

$$A^T A = \begin{pmatrix} 8 & 8 & 0 \\ 8 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

hence the characteristic polynomial of $A^T A$ is given by

$$\begin{aligned}
\det(A^T A - \lambda i) =& \begin{vmatrix} 8-\lambda & 8 & 0 \\ 8 & 8-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} \\
=& -\lambda(\lambda - 1)(\lambda - 16).
\end{aligned}$$

So the eigenvalues of $A^T A$ is $\lambda_1 = 16$, $\lambda_2 = 1$ and $\lambda_3 = 0$. The corresponding eigenvectors are $\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. Since $A^T = A$, we have $\vec{u}_i = \vec{v}_i$ for $1 \le i \le 3$. Therefore, $U = V = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.