MATH3290 Mathematical Modeling 2023/2024
Assignment 3
Suggested Solutions

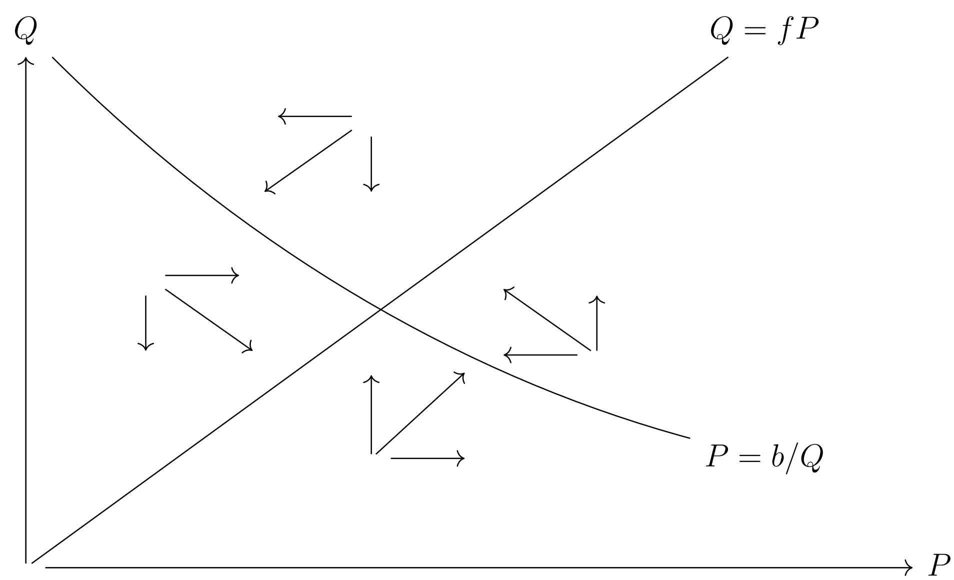
1. (a) The equilibrium points occur when $dP/dt = dQ/dt = 0$. It is clear that $dP/dt = 0$ if and only if $P = 0$ or $P = b/Q$, and $dQ/dt = 0$ if and only if $Q = 0$ or $Q = fP$. Since dP/dt is not well defined when $Q = 0$, it is clear that the only equilibrium points occurs when $P = b/Q$ and $Q = fP$. These conditions imply

$$P = \frac{b}{fP} \implies P^2 = \frac{b}{f} \implies P = \pm\sqrt{\frac{b}{f}} \tag{1}$$

Since P is positive, we must reject $P = -\sqrt{b/f}$. It follows the only equilibrium point in this case is $(P, Q) = (\sqrt{b/f}, \sqrt{bf})$.

(b) By substituting the given values, it follows the equilibrium point is $(P, Q) = (\sqrt{2000/3}, \sqrt{600000}) \approx (25.82, 774.60)$.

(c) By considering the signs of dP/dt and dQ/dt in each region, which are divided by the curves $Q = fP$ and $P = bQ$, we arrive at the following phase plane:



Note that the intersection of the curves is the equilibrium point found in part (a). We cannot determine if the equilibrium point is stable, as the phase lines rotate around the equilibrium point (i.e. we cannot determine if solutions while spiral outward, inward, or stay periodic around the equilibrium point).

2. (a) The differential equation can be solved using separation of variables. We have that

$$\frac{dP}{P(M-P)(P-m)} = rdt \tag{2}$$

Partial fraction decomposition yields

$$\frac{1}{P(M-P)(P-m)} = \frac{1}{M-m}\left(\frac{1}{P(P-m)} + \frac{1}{P(M-P)}\right) \tag{3}$$

We can further see that

$$\frac{1}{P(P-m)} = \frac{1}{m}\left(\frac{1}{P-m} - \frac{1}{P}\right) \; ; \; \frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right) \tag{4}$$

Hence, substituting into (2) we have

$$\frac{1}{m(M-m)}\int \frac{1}{P-m} - \frac{1}{P}\, dP + \frac{1}{M(M-m)}\int \frac{1}{P} + \frac{1}{M-P}\, dP = \int r\, dt \tag{5}$$

After integrating, we see that we have

$$\frac{\ln|P-m| - \ln P}{m(M-m)} + \frac{\ln P - \ln|M-P|}{M(M-m)} = rt + C \tag{6}$$

Note that the absolute value is omitted for the $\ln P$ terms as P must be positive. We may also rewrite as

$$P^{m-M}|P-m|^{M}|M-P|^{-m} = Ae^{Mmr(M-m)T} \tag{7}$$

(where A is a constant) for simplicity.

(b) By considering the differential equation, we see that $dP/dt < 0$ in this case. Hence, P must approach M as $t \to \infty$.

(c) Similar to part (b), we can split into two cases. If $m < P < M$ then $dP/dt > 0$ and P approaches M. If $P < m$ then $dP/dt < 0$ and P approaches 0.

(d) The equilibrium points are $P = 0, m, M$. The steady state value depends on the initial value in the following way, which was proved in the previous parts: If $P_0 \geq m$, then $P \to M$; if $P_0 < m$, then $P \to 0$.

3. (a) Let us consider the system as a whole. There is no net force in the system as the force applied on the rocket is of equal and opposite magnitude to the force applied on the propellant mass. The total change in momentum in a small time Δt is given by $(m-\Delta m_p)(v+\Delta v)-(\Delta m_p)u-mv$ (as u is of the opposite direction to v). Hence, dividing by Δt, ignoring the $\Delta m_p \Delta v$ term as it is of higher order, and taking $\Delta t \to 0$ yields

$$\frac{d}{dt}(\text{momentum}) = -v\frac{dm_p}{dt} + m\frac{dv}{dt} - u\frac{dm_p}{dt} \tag{8}$$

By the discussion about the net force, the above quantity is equal to 0. Moroever, $dm_p/dt = -dm/dt$, so

$$0 = c\frac{dm}{dt} + m\frac{dv}{dt} \tag{9}$$

Rearranging yields

$$\frac{dv}{dt} = -\frac{c}{m}\frac{dm}{dt} \tag{10}$$

as required.

(b) Multiplying both side by dt/dm (this is justified as dm/dt is non-zero) yields

$$\frac{dv}{dm} = \frac{-c}{m} \tag{11}$$

Separation of variables directly yields

$$\ln m = -\frac{1}{c}v + C \tag{12}$$

We are given that at $t = 0$, $v = 0$ and $m = M + P$. Hence, $C = \ln(M + P)$ and we have that

$$v = -c\ln m - \ln(M+P) = -c\ln\left[\frac{m}{M+P}\right] \tag{13}$$

as required.

(c) The case where all fuel is burned corresponds to when $m = (1-\varepsilon)M + P$. Then, we have that

$$v_f = -c\ln\left[\frac{(1-\varepsilon)M+P}{M+P}\right] = -c\ln\left[1-\frac{\varepsilon M}{M+P}\right] = -c\ln\left[1-\frac{1}{1+\beta}\right] \tag{14}$$

(d) Plugging the values into part (c) we see that $v_f \approx 4.71$ km/s.