MATH3290 Mathematical Modeling 2023/2024
Assignment 3
Due: 5pm, April 16th

Note: Submit your assignment via Blackboard. Late submissions are not allowed.

1. Consider the following economic model: Let P be the price of a single item on the market. Let Q be the quantity of the item available on the market. Both P and Q are functions of time. If we consider price and quantity as two interacting species, the following model might be proposed as follows

$$\begin{aligned}\frac{dP}{dt} &= aP(b/Q - P),\\ \frac{dQ}{dt} &= cQ(fP - Q),\end{aligned}$$

where a, b, c and f are positive constants.

   (a) Find the equilibrium points of this system in terms of the constants a, b, c and f.

   (b) If $a = 1$, $b = 20,000$, $c = 1$ and $f = 30$, calculate the equilibrium points of this system using the result of (a).

   (c) Perform a graphical stability analysis to determine what will happen to the levels of P and Q as time increase. Also, classify each equilibrium point with respect to its stability, if possible. If a point cannot be readily classified, explain the reason.

2. The fish and game department in a certain state is planning to issue hunting permits to control the deer population (one deer per permit). It is known that if the deer population falls below a certain level m, the deer will become extinct. It is also known that if the deer population rises above the carrying capacity M, the population will decrease back to M through disease and malnutrition. Assume that P is the population of the deer and r is a positive constant of proportionality. The model can be formulated as follows

$$\frac{dP}{dt} = rP(M - P)(P - m).$$

   (a) Write down the explicit formula for the population P in terms of r, m, M and the integral constant (if necessary).

   (b) Show that if $P > M$ for all t, then we have

$$\lim_{t\to\infty} P(t) = M.$$

   (c) What happens if $P < M$ for all t?

(d) What are the equilibrium points of the model? Explain the dependence of the steady-state value of P on the initial values of P.

3. Consider launching a satellite into orbit using a single-stage rocket. The rocket is continuously losing mass, which is being propelled away from it at significant speeds. We are interested in predicting the maximum speed the rocket can attain.

(a) Assume the rocket of mass m is moving with speed v. In a small increment of time Δt it loses a small mass Δm_p, which leaves the rocket with speed u in a direction opposite to v. Here, Δm_p is the small propellant mass. The resulting speed of the rocket is $v + \Delta v$. Neglect all external forces (gravity, atmospheric drag, etc.) and assume Newton's second law of motion:

$$\text{force} = \frac{d}{dt}(\text{momentum of system})$$

where momentum is mass times velocity. Derive the model

$$\frac{dv}{dt} = \left(\frac{-c}{m}\right)\frac{dm}{dt}$$

where $c = u + v$ is the relative exhaust speed (the speed of the burnt gases relative to the rocket).

(b) Assume that initially, at time $t = 0$, the velocity $v = 0$ and the mass of the rocket is $m = M + P$, where P is the mass of the payload satellite and $M = \varepsilon M + (1 - \varepsilon)M$ $(0 < \varepsilon < 1)$ is the initial fuel mass εM plus the mass $(1 - \varepsilon)M$ of the rocket casings and instruments. Solve the model in part(i) to obtain the speed

$$v = -c \ln\left[\frac{m}{M + P}\right].$$

(c) Show that when all fuel is burned, the speed of the rocket is given by

$$v_f = -c \ln\left[1 - \frac{\varepsilon}{1 + \beta}\right]$$

where $\beta = P/M$ is the ratio of the payload mass to the rocket mass.

(d) Find v_f if $c = 3$ km/sec, $\varepsilon = 0.8$ and $\beta = 0.01$. (These are typical values in satellite launchings.)