MATH3290 Mathematical Modeling (2023-24)
Assignment 1
Suggested Solutions

1. Consider the data sets in Table 1

x	1	2	3	4	5	6	7
y_1	7	15	33	61	99	147	205
y_2	4.5	20	90	403	1,808	8,130	36,316

Table 1: Data set for Problem 1.

(a) For (x, y_1), construct a divided difference table. What conclusions can you make about y_1? Would you use a low-order polynomial as an empirical model? If so, what order?

(b) For (x, y_2), construct a divided difference table. Would you use a low-order polynomial as an empirical model? If not, give the reason.

***Solution* :**

(a) The divided difference table for (x, y_1) is Table 2. Note that the second divided differences are essentially constant and that the third divided differences are zero.

x	y_1	Δ	Δ^2	Δ^3
1	7	8	5	0
2	15	18	5	0
3	33	28	5	0
4	61	38	5	0
5	99	48	5	
6	147	58		
7	205			

Table 2: Data set for Problem 1(a).

(b) From Table 3, we can find that the sixth divided differences are not zero. So we can not use a low-order polynomial as an empirical model.

x	y_2	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1	4.5	15.5	27.25	31.41666667	27.52083333	19.29583333	10.87569444
2	20	70	121.5	141.5	124	84.55	
3	90	313	546	637.5	546.75		
4	403	1405	2458.5	2824.5			
5	1808	6322	10932				
6	8130	28186					
7	36316						

Table 3: Data set for Problem 1(b).

2. The following data were obtained for the growth of a sheep population introduced into a new environment on the island of Tasmania.

Year	1814	1824	1834	1844	1854	1864
Population	125	275	830	1200	1750	1650

Table 4: Data set for problem 1.

(a) Plot the change in population versus year. Is there a trend?

(b) Formulate a discrete dynamical system model. Use the least-squares criterion to find the model parameter.

(c) Predict the sheep population in the year 1869.

Solution **:**

(a) The change was shown in Figure 1. Thus, we may assume it is parabolic.

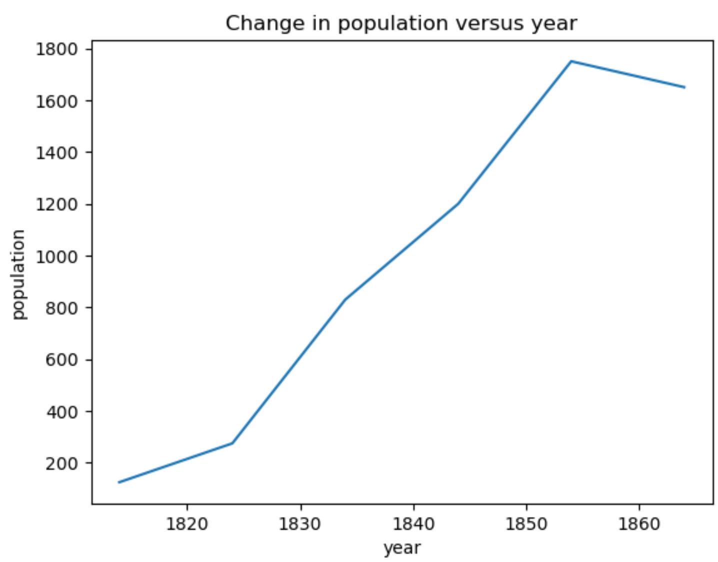

Figure 1: difference of population of sheep.

(b) Replace the year-variable n with variable $m = \frac{n-1804}{10}$, so m is from 1 to 6. From (a) we know the difference is a parabolic:

$$P_{m+1} - P_m = \Delta P_m \approx am^2 + bm + c, \quad m = 1, 2, \cdots, 5$$

where P_m is sheep population at year $n(n = 10m + 1804)$; a, b, c are parameters¡£ Let $t_m = \Delta P_m$, it's easy to see $(t_1, \cdots, t_5) = (150, 555, 370, 550, -100)^T$. Using the Least-Squares criterion, that is to minimize

$$S(a, b, c) = \sum_{m=1}^{5} (am^2 + bm + c - t_m)^2$$

In order to solve this minimum problem, taking partial derivatives

$$\begin{aligned} 0 &= \frac{\partial S}{\partial a} = \sum_{m=1}^{5} 2m^2(am^2 + bm + c - t_m) \\ 0 &= \frac{\partial S}{\partial b} = \sum_{m=1}^{5} 2m(am^2 + bm + c - t_m) \\ 0 &= \frac{\partial S}{\partial c} = \sum_{m=1}^{5} 2(am^2 + bm + c - t_m) \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{m=1}^{5} m^2 t_m &= a\sum_{m=1}^{5} m^4 + b\sum_{m=1}^{5} m^3 + c\sum_{m=1}^{5} m^2 \\ \sum_{m=1}^{5} m t_m &= a\sum_{m=1}^{5} m^3 + b\sum_{m=1}^{5} m^2 + c\sum_{m=1}^{5} m \\ \sum_{m=1}^{5} t_m &= a\sum_{m=1}^{5} m^2 + b\sum_{m=1}^{5} m + c\sum_{m=1}^{5} 1 \end{aligned}$$

That is

$$\begin{aligned} 12000 &= 979a + 225b + 55c \\ 4047 &= 225a + 55b + 15c \\ 1525 &= 55a + 15b + 5c \end{aligned}$$

So $(a, b, c) = (-124.6429, 697.3571, -416.0000)$
So the model is $\Delta P_m = -124.6429m^2 + 697.3571m - 416.0000$.

(c) According to (b) we have $\Delta P_m = -124.6429m^2 + 697.3571m - 416.0000$. When $n = 1869$, we have $m + 1 = 6.5$, then we can get $\Delta P_{5.5} = -350.9821$. If we approximate $P_{5.5}$ by $P_{5.5} = \frac{1}{2}(P_6 + P_5) = \frac{1}{2}(1750 + 1650) = 1700$, then $P_{6.5} = P_{5.5} + \Delta P_{5.5} = 1700 - 350.9821 = 1349.0179$.

3. **(Markov process)** A certain protein molecule can have three configurations which are denoted as C_1, C_2 and C_3. Every second, a protein molecule can make a transition from one configuration to another configuration with the following probabilities:

$$\begin{array}{lll} P(C_1 \to C_1) = 0.3 & P(C_1 \to C_2) = 0.2 & P(C_1 \to C_3) = 0.5 \\ P(C_2 \to C_1) = 0.3 & P(C_2 \to C_2) = 0.5 & P(C_2 \to C_3) = 0.2 \\ P(C_3 \to C_1) = 0.4 & P(C_3 \to C_2) = 0.2 & P(C_3 \to C_1) = 0.4 \end{array}$$

The configuration transition are demonstrated in Figure 2. (For example, the molecule will transit from C_1 to C_2 with probability 0.2.)

Consider a living body with a fixed number of protein molecules. We let C_i^n ($i = 1,2,3; n = 0,1,2,\dots$) be the percentage of molecules that are in configuration C_i ($i = 1,2,3$) at the end of the n-th second.

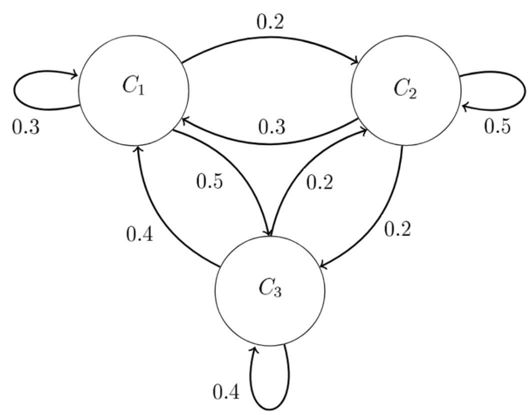

Figure 2: An illustration of the configuration transition process for Problem 3.

(a) Formulate a model for C_i^n using a system of difference equations.

(b) Find the equilibrium point, and determine its stability.

(c) Consider the three initial conditions in Table 5. Compute C_1^5, C_2^5, C_3^5 for each case. Does the long term behaviour sensitive to the initial condition?

Percentage	C_1^0	C_2^0	C_3^0
Case A	0	0	1
Case B	0	0.5	0.5
Case C	0.2	0.2	0.6

Table 5: Data set for Problem 3.

Solution **:**

(a) Formulating the system of difference equations, we have the following dynamical system:

$$\begin{aligned} C_1^{n+1} &= 0.3C_1^n + 0.3C_2^n + 0.4C_3^n \\ C_2^{n+1} &= 0.2C_1^n + 0.5C_2^n + 0.2C_3^n \\ C_3^{n+1} &= 0.5C_1^n + 0.2C_2^n + 0.4C_3^n \end{aligned}$$

Then $\begin{pmatrix} C_1^{n+1} \\ C_2^{n+1} \\ C_3^{n+1} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.2 & 0.5 & 0.2 \\ 0.5 & 0.2 & 0.4 \end{pmatrix} \begin{pmatrix} C_1^n \\ C_2^n \\ C_3^n \end{pmatrix}$. Denote this equations as $C_{n+1} = AC_n$.

(b) Suppose (C_1^*, C_2^*, C_3^*) are the equilibrium values of this system, then we must have $C_1^* = C_1^{n+1} = C_1^n, C_2^* = C_2^{n+1} = C_2^n, C_3^* = C_3^{n+1} = C_3^n$. Substituting into the dynamical system yields

$$\begin{aligned} 0.7C_1^* &= 0.3C_2^* + 0.4C_3^* \\ 0.5C_2^* &= 0.2C_1^* + 0.2C_3^* \\ 0.6C_3^* &= 0.5C_1^* + 0.2C_2^*. \end{aligned}$$

There are an infinite number of solutions to this system of equations. Suppose $C_3^* = 1$, we find the dynamical system is satisfied when $C_1^* = 0.8966, C_2^* = 0.7586$. Keep this proportion and let C_1^*, C_2^*, C_3^* satisfy $C_1^* + C_2^* + C_3^* = 1$, we can get the equilibrium point $C^* = (0.3377, 0.2857, 0.3766)$.

To determine the stability of this equilibrium point, first we calculate the eigenvalues of A. They are $\lambda_1 = 1, \lambda_2 = -0.1, \lambda_3 = 0.3$. And the corrsponding eigenvectors are

$$X_1 = \begin{pmatrix} -0.5812 \\ -0.4918 \\ -0.6483 \end{pmatrix}, X_2 = \begin{pmatrix} -0.7071 \\ 0.0000 \\ 0.7071 \end{pmatrix}, X_3 = \begin{pmatrix} 0.1961 \\ -0.7845 \\ 0.5883 \end{pmatrix}.$$

Since X_1, X_2, X_3 are linearly independent, then C_0 can be written as

$$C_0 = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3.$$

Then

$$\begin{aligned} C_n &= A^n C_0 = A^n(\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3) \\ &= A^{n-1}(\alpha_1 \lambda_1 X_1 + \alpha_2 \lambda_2 X_2 + \alpha_3 \lambda_3 X_3) \\ &\cdots \\ &= \alpha_1 \lambda_1^n X_1 + \alpha_2 \lambda_2^n X_2 + \alpha_3 \lambda_3^n X_3. \end{aligned}$$

Since $|\lambda_2| < 1, |\lambda_3| < 1$, then

$$\lim_{n \to +\infty} \lambda_2^n = 0, \lim_{n \to +\infty} \lambda_3^n = 0.$$

Hence,

$$\lim_{n \to +\infty} C_n = \alpha_1 X_1.$$

So this equilibrium point is stable and independent of the initial condition.

(c) Use the equations in (a), it's easy to compute C_1^5, C_2^5, C_3^5 for each case. The tendency in each case can be seen in Figure 3-5. It's esay to see that the long term behaviour does not sensitive to the initial condition.

Percentage	C_1^5	C_2^5	C_3^5
Case A	0.3378	0.2850	0.3771
Case B	0.3375	0.2862	0.3762
Case C	0.3377	0.2855	0.3768

Table 6: Data set for Problem 3.

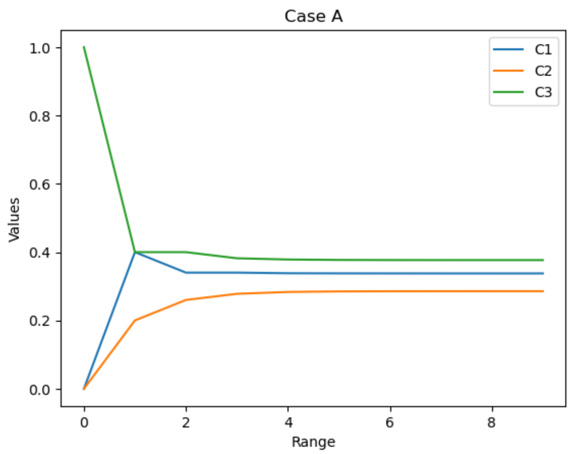

Figure 3: Case A of Problem 3(c)

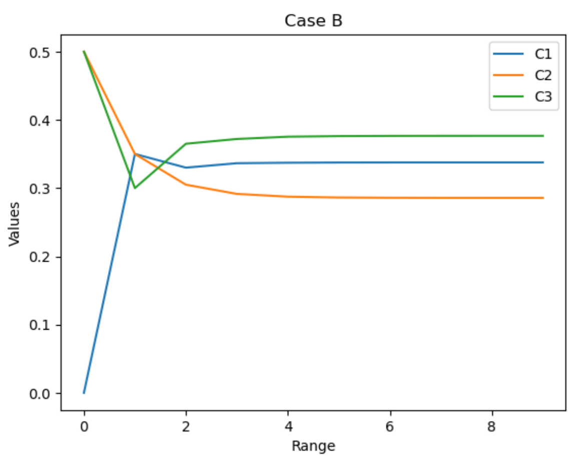

Figure 4: Case B of Problem 3(c)

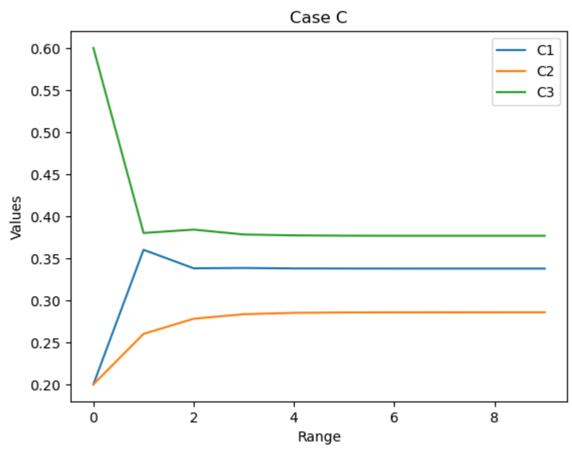

Figure 5: Case C of Problem 3(c)