MATH 3290 Mathematical Modeling

Chapter 7: Optimization of Discrete Models

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Course webpage

https://www.math.cuhk.edu.hk/course/2324/math3290

SCAN ME

Overview of optimization

The general form of optimization problem: find X^* which

$$\text{optimizes} \quad f(X)$$

subject to the following conditions

$$g_i(X) \geq b_i, \qquad i = 1, 2, \ldots, m.$$

- $f(X)$ is called the objective function;
- $g_i(X) \geq b_i$ are the constraints;
- $X = (X_1, X_2, \ldots, X_n)$ are the decision variables;
- optimization can be maximization or minimization.

We consider Linear Programming (LP) in this chapter, that is, both $f(X)$ and $g_i(X)$ are linear functions of X. When X are integers, it is called integer programming.

Example 1: Chebyshev criterion

Consider a data set (x_i, y_i), $i = 1, 2, \ldots, m$.

We fit the model function $y = ax + b$ by the Chebyshev criterion.

We find a and b which minimize

$$\max_{i=1,\ldots,m} |y_i - f(x_i; a, b)|$$

To transform the above problem as a LP problem, we introduce a new variable $r = \max_i |y_i - f(x_i; a, b)| = \max_i |y_i - ax_i - b|$.

Then

$$r \geq |y_i - ax_i - b|, \qquad i = 1, 2, \ldots, m,$$

which is equivalent to

$$r \geq y_i - ax_i - b, \quad -r \leq y_i - ax_i - b, \qquad i = 1, 2, \ldots, m.$$

Combining above, the problem can be formulated as

$$\text{minimize} \quad r$$

subject to

$$r - (y_i - ax_i - b) \geq 0, \quad r + (y_i - ax_i - b) \geq 0, \qquad i = 1, 2, \ldots, m.$$

Note:

- the decision variables are r, a, and b;
- the objective function $f(r, a, b) = r$, which is linear;
- there are $2m$ constraints, they are all linear functions of r, a, and b.

Example 2: Carpenter's problem

A carpenter makes tables and bookcases.

- Net profits of \$25 per table, and \$30 per bookcase.
- The carpenter has 690 units of wood, and 120 units of labor.
- Each table requires 20 units of wood and 5 units of labor.
- Each bookcase requires 30 units of wood and 4 units of labor.

The carpenter is trying to determine how many of each he should make in order to maximize his profit.

Recall assumptions:

- Net profits of \$25 per table, and \$30 per bookcase.
- The carpenter has 690 units of wood, and 120 units of labor.
- Each table requires 20 units of wood and 5 units of labor.
- Each bookcase requires 30 units of wood and 4 units of labor.

Let x_1 and x_2 be numbers of tables and bookcases. We can then formulate the following

$$\text{maximize} \quad 25x_1 + 30x_2$$

subject to

$$\begin{aligned} 20x_1 + 30x_2 &\leq 690, \\ 5x_1 + 4x_2 &\leq 120, \end{aligned}$$

where $x_1 \geq 0$ and $x_2 \geq 0$. (Note that generally we need x_1 and x_2 to be integers.)

General form of LP

We will consider the following form of LP

$$\text{maximize} \quad c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to the inequality constraints

$$\begin{aligned} g_{11}x_1 + g_{12}x_2 + \cdots + g_{1n}x_n &\leq b_1, \\ g_{21}x_1 + g_{22}x_2 + \cdots + g_{2n}x_n &\leq b_2, \\ &\vdots \\ g_{m1}x_1 + g_{m2}x_2 + \cdots + g_{mn}x_n &\leq b_m, \end{aligned}$$

where $x_1, x_2, \ldots, x_n \geq 0$ (non-negativity conditions).

Other LP problems can be written in this form.
$(x \in \mathbb{R} \Leftrightarrow x = x_1 - x_2, x_1, x_2 \geq 0.)$

Solve LP: geometric method

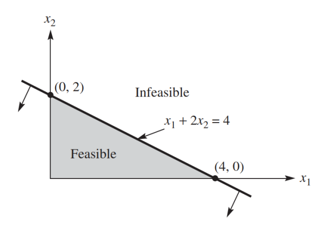
Feasible region = the region defined by the inequality constraints.

LP = maximize objective function over the feasible region.

Example: visualize the feasible region defined by

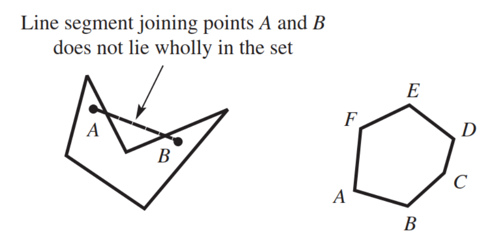
$$x_1 + 2x_2 \leq 4, \qquad x_1 \geq 0, \quad x_2 \geq 0.$$

- Conditions $x_1, x_2 \geq 0$ show the first quadrant contains the feasible region.
- The line $x_1 + 2x_2 = 4$ divides the first quadrant into two regions, and select one point (e.g. $(0,0)$) from each region to determine which one is feasible.



Important facts about feasible regions:

- The feasible region of a LP problem is a convex set (for every pair of points in a convex set, the line segment joining them lies in the set).



Left: non-convex. Right: convex.

- A solution of a LP problem must be at one of the corner (extreme) points. (see points A-F above)

$$\text{Maximize} \quad f(x_1, x_2) = 25x_1 + 30x_2$$

subject to

$$20x_1 + 30x_2 \leq 690 \qquad \text{(constraint 1)},$$
$$5x_1 + 4x_2 \leq 120 \qquad \text{(constraint 2)},$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

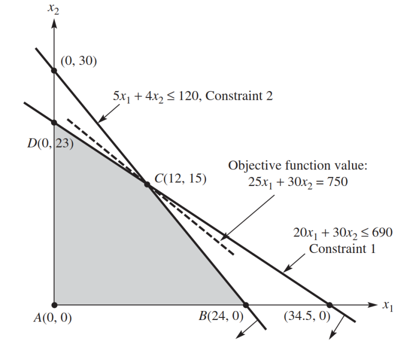
Forming the feasible region:

- for constraint 1, consider

$$20x_1 + 30x_2 = 690;$$

- for constraint 2, consider

$$5x_1 + 4x_2 = 120.$$



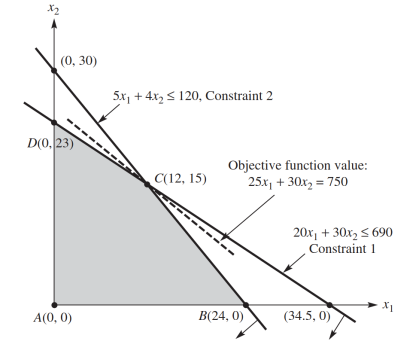
Then look at corner points (there are 4) of the feasible region:

Use the objective function

$$f = 25x_1 + 30x_2$$

to compute f at extreme points.

Extreme point	Objective function value
A (0, 0)	\$0
B (24, 0)	600
C (12, 15)	750
D (0, 23)	690

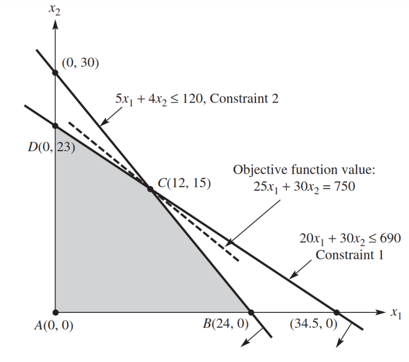


We see that the objective function is maximized at point C.

Hence, an optimal solution is $x_1 = 12, x_2 = 15$, and the optimal value of f is 750.

An important observation:

Consider the line defined by $f(x_1, x_2) = 25x_1 + 30x_2 = 750$.



We see that it intersects the feasible region only at the optimal solution $(x_1, x_2) = (12, 15)$. The LP problem has a unique solution.

Example: model fitting by the Chebyshev criterion.

Consider fitting the model function $y = cx$ to the data

x	1	2	3
y	2	5	8

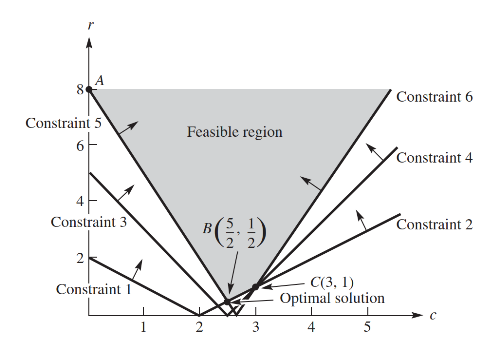
From earlier discussions, we obtain the LP problem

$$\text{minimize} \quad f(c, r) = r$$

subject to

$$\begin{aligned} r - (2 - c) &\geq 0, & r + (2 - c) &\geq 0, \\ r - (5 - 2c) &\geq 0, & r + (5 - 2c) &\geq 0, \\ r - (8 - 3c) &\geq 0, & r + (8 - 3c) &\geq 0, \end{aligned}$$

where $r \geq 0$. It is also not harmful to assume $c \geq 0$.



$$
\begin{aligned}
r - (2 - c) &\ge 0 && \text{(constraint 1)}\\
r + (2 - c) &\ge 0 && \text{(constraint 2)}\\
r - (5 - 2c) &\ge 0 && \text{(constraint 3)}\\
r + (5 - 2c) &\ge 0 && \text{(constraint 4)}\\
r - (8 - 3c) &\ge 0 && \text{(constraint 5)}\\
r + (8 - 3c) &\ge 0 && \text{(constraint 6)}
\end{aligned}
$$

Note that the extreme point B is the intersection of lines 2 and 5.

$$r + (2 - c) = 0, \qquad r - (8 - 3c) = 0.$$

Solving it, we have $c = 5/2$ and $r = 1/2$.

Coordinates of A and C are found similarly.

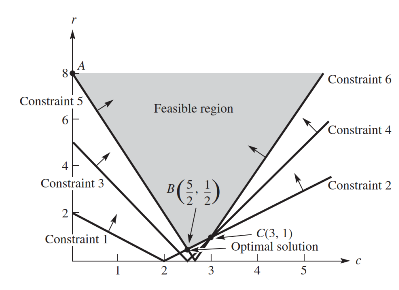
Then look at corner points (there are 3) of the feasible region:

Use the objective function

$$f(c, r) = r$$

to compute f at extreme points.

Extreme point	Objective function value
(c, r)	$f(r) = r$
A	8
B	$\frac{1}{2}$
C	1



We see that the objective function is minimized at point B.

The solution is $c = 5/2$, $r = 1/2$, and the optimal value of f is $1/2$.

Hence, the model function is $y = 5x/2$.

Solve LP: Algebraic method

Main idea:

1. Find all intersection points defined by constraints.
2. Determine if they are feasible.
3. Evaluate values of the objective function at extreme points.
4. Choose the point which gives the optimal objective function value.

Next, we illustrate this by an example.

Example: consider again carpenter's problem

$$\text{maximize} \quad f(x_1, x_2) = 25x_1 + 30x_2$$

subject to

$$\begin{aligned} 20x_1 + 30x_2 &\leq 690, \\ 5x_1 + 4x_2 &\leq 120, \end{aligned}$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

Then, we introduce slack variables $y_1, y_2 \geq 0$ so that

$$\begin{aligned} 20x_1 + 30x_2 + y_1 &= 690, \\ 5x_1 + 4x_2 + y_2 &= 120. \end{aligned}$$

Step 1: we need to find all intersection points.

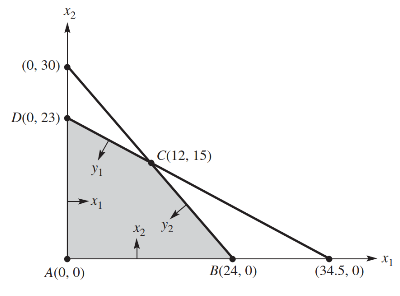
We have

$$20x_1 + 30x_2 + y_1 = 690,$$
$$5x_1 + 4x_2 + y_2 = 120,$$

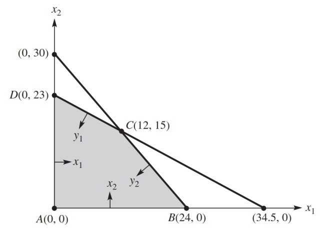
where x_1, x_2, y_1, and $y_2 \geq 0$.

To find an intersection point, we set 2 of $\{x_1, x_2, y_1, y_2\}$ to zero then solve the other 2 unknowns by the above equations.

Hence, there are totally 6 intersection points.

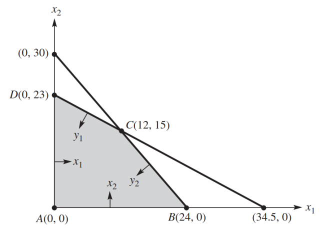


$$20x_1 + 30x_2 + y_1 = 690,$$
$$5x_1 + 4x_2 + y_2 = 120.$$



set to zero	solution of the 2 equations	intersection point
$x_1 = 0, x_2 = 0$	$y_1 = 690, y_2 = 120$	$A(0, 0)$
$x_1 = 0, y_1 = 0$	$x_2 = 23, y_2 = 28$	$D(0, 23)$
$x_1 = 0, y_2 = 0$	$x_2 = 30, y_1 = -210$	$(0, 30)$
$y_1 = 0, y_2 = 0$	$x_1 = 12, y_2 = 15$	$C(12, 15)$
$x_2 = 0, y_1 = 0$	$x_1 = 34.5, y_2 = -52.5$	$(34.5, 0)$
$x_2 = 0, y_2 = 0$	$x_1 = 24, y_1 = 210$	$B(24, 0)$

Step 2: determine which point is feasible.



Negative values of $\{x_1, x_2, y_1, y_2\}$ imply infeasible.

set to zero	solution of the 2 equations	intersection point	feasible
$x_1 = 0, x_2 = 0$	$y_1 = 690, y_2 = 120$	$A(0, 0)$	Y
$x_1 = 0, y_1 = 0$	$x_2 = 23, y_2 = 28$	$D(0, 23)$	Y
$x_1 = 0, y_2 = 0$	$x_2 = 30, y_1 = -210$	$(0, 30)$	N
$y_1 = 0, y_2 = 0$	$x_1 = 12, y_2 = 15$	$C(12, 15)$	Y
$x_2 = 0, y_1 = 0$	$x_1 = 34.5, y_2 = -52.5$	$(34.5, 0)$	N
$x_2 = 0, y_2 = 0$	$x_1 = 24, y_1 = 210$	$B(24, 0)$	Y

Step 3: evaluate objective function at feasible points.

$$f(x_1, x_2) = 25x_1 + 30x_2.$$

Extreme point	Objective function value
A (0, 0)	$0
B (24, 0)	600
C (12, 15)	750
D (0, 23)	690

Step 4: find the point giving the optimal value.

The point C gives the maximum value of f.

Hence, the optimal solution $x_1 = 12, x_2 = 15$.

A big disadvantage of this algebraic method—too costly.

Consider a LP problem with m decision variables and n constraints.

Then for each constraint, we introduce a new slack variable. Hence, there are $m + n$ variables.

We set m of them to zero and solve the other n.

There are totally $\dfrac{(m+n)!}{m!n!}$ intersection points.

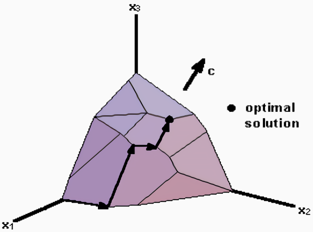
e.g. if $m = 14, n = 14$, there are 40,116,600 intersection points!

We have Dantzig's simplex method, which shares a similar idea but no need to compute all intersection points.

Solve LP: Simplex method

Overview:

1. start at an intersection point;
2. check if the point gives an optimal value;
3. if not, move to the next feasible intersection point that gives a better value, then go back to step 2.



In the following, we give concrete meaning of optimality test and feasibility test.

$$\text{Maximize} \quad c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to

$$\begin{aligned} g_{11}x_1 + g_{12}x_2 + \cdots + g_{1n}x_n + y_1 &= b_1 \\ &\vdots \\ g_{m1}x_1 + g_{m2}x_2 + \cdots + g_{mn}x_n + y_m &= b_m \end{aligned}$$

where $x_1, x_2, \ldots, x_n \geq 0$ and $y_1, \ldots, y_m \geq 0$.

- $x_1, x_2, \ldots, x_n$ are decision variables;
- $y_1, y_2, \ldots, y_m$ are slack variables;
- an intersection point is obtained when n of the variables are set to zero, these are called independent variables;
- the values of the other m variables are obtained by solving the above system, these are called dependent variables.

Steps of Simplex Method

1. **Initialize**: starts at an extreme point, usually the origin $(x_1, x_2, \dots, x_n) = (0, 0, \dots, 0)$ if $b_1, b_2, \dots, b_m \geq 0$.
2. **Optimality test**: determine if there is an adjacent intersection point that improves the value of the objective function.
   - Mathematically, one of independent variables (which is currently zero) should become dependent (thus non-zero), entering the dependent set.
3. **Feasibility test**: to find a new neighboring feasible intersection point.
   - From step 2, we need one more independent variable.
   - One of the current dependent variables should be changed to independent, leaving the dependent set.
4. **Pivot**: solve the resulting linear system.
5. **Repeat**: go back to step 2.

$$\text{Maximize} \quad f(x_1, x_2) = 25x_1 + 30x_2$$

subject to

$$20x_1 + 30x_2 \leq 690,$$
$$5x_1 + 4x_2 \leq 120.$$

Note that we can write the objective function as

$$z = 25x_1 + 30x_2 \geq 0,$$

because $(x_1, x_2) = (0, 0)$ is a feasible point.

Then, we introduce slack variables y_1, y_2, and $z \geq 0$ so that

$$\begin{aligned} 20x_1 + 30x_2 + y_1 &= 690, \\ 5x_1 + 4x_2 + y_2 &= 120, \\ -25x_1 - 30x_2 + z &= 0. \end{aligned}$$

The last equation comes from the objective function.

Step **1**: initialize,

$$\begin{aligned} 20x_1 + 30x_2 + y_1 &= 690, \\ 5x_1 + 4x_2 + y_2 &= 120, \\ -25x_1 - 30x_2 + z &= 0. \end{aligned}$$

Set $x_1 = x_2 = 0$. Then solving the first two equations
$\Rightarrow y_1 = 690, y_2 = 120$.

Moreover, solving the last equation, $z = 0$.

- The independent set = $\{x_1, x_2\}$.
- The dependent set = $\{y_1, y_2, z\}$.
- The current extreme point = $(x_1, x_2) = (0, 0)$.
- The current value of the objective function $z = 0$.

Step **2**: optimality test, choosing entering variable

$$20x_1 + 30x_2 + y_1 = 690,$$
$$5x_1 + 4x_2 + y_2 = 120,$$
$$-25x_1 - 30x_2 + z = 0.$$

Currently, the independent set is $\{x_1, x_2\}$.

From the last equation, the coefficients of x_1 and x_2 are negative. This means if one of them becomes positive, then the value of objective function z becomes positive (improved).

Hence, one of x_1 and x_2 should enter the dependent set.

As a rule, choose the one with the most negative coefficient.

In this case, x_2 is the entering variable.

Step 3: feasibility test, choosing the leaving variable,

$$20x_1 + 30x_2 + y_1 = 690,$$
$$5x_1 + 4x_2 + y_2 = 120.$$

Currently, the dependent set is $\{y_1, y_2, z\}$. One of $\{y_1, y_2\}$ is leaving.

Dividing the right-hand side by the coefficient of x_2 (the entering variable).

$$r_1 = \frac{690}{30} = 23, \qquad r_2 = \frac{120}{4} = 30.$$

Note r_1 is the value of x_2 when $y_1 = 0$, r_2 value of x_2 if $y_2 = 0$.

As a rule, we choose the leaving variable with the smallest positive ratio.

In this case, y_1 is chosen as the leaving variable.

Step 4: pivot, solve the resulting linear system,

$$\begin{aligned} 20x_1 + 30x_2 + y_1 &= 690, \\ 5x_1 + 4x_2 + y_2 &= 120, \\ -25x_1 - 30x_2 + z &= 0. \end{aligned}$$

$$\text{the independent set} = \{x_1, y_1\}, \quad \text{the dependent set} = \{x_2, y_2, z\}.$$

Setting $x_1 = y_1 = 0$ in the first two equations

$$30x_2 = 690, \qquad 4x_2 + y_2 = 120.$$

We have $x_2 = 23$ and $y_2 = 28$.

Hence, the current extreme point is $(x_1, x_2) = (0, 23)$, and the current value of the objective function is $z = 690$.

Step 5: repeat the above.

Tableau format

Consider the same example, we have

$$20x_1 + 30x_2 + y_1 = 690,$$
$$5x_1 + 4x_2 + y_2 = 120,$$
$$-25x_1 - 30x_2 + z = 0.$$

Step 1: initialize, it is more convenient to set up a tableau format:

x_1	x_2	y_1	y_2	z	RHS
20	30	1	0	0	$690\ (= y_1)$
5	4	0	1	0	$120\ (= y_2)$
-25	-30 (circled)	0	0	1	$0\ (= z)$

Dependent variables: $\{y_1, y_2, z\}$
Independent variables: $x_1 = x_2 = 0$
Extreme point: $(x_1, x_2) = (0, 0)$
Value of objective function: $z = 0$

Entering variable ↓ x_2

x_1	x_2	y_1	y_2	z	RHS	Ratio
20	30	1	0	0	690	(23) (= 690/30) ← Exiting variable
5	4	0	1	0	120	30 (= 120/4)
−25	(−30)	0	0	1	0	*

Step 2: optimality, choosing the entering variable (the variable with most negative coefficient x_2).

Step 3: feasibility, choosing the leaving variable (the variable with the smallest positive ratio y_1).

Entering variable: x_2

x_1	x_2	y_1	y_2	z	RHS	Ratio
20	30	1	0	0	690	(23) (= 690/30) ← Exiting variable
5	4	0	1	0	120	30 (= 120/4)
−25	(−30)	0	0	1	0	*

Step 4: pivot, row operations with respect to the column containing entering variable.

x_1	x_2	y_1	y_2	z	RHS
0.66667	1	0.03333	0	0	23 $(= x_2)$
2.33333	0	−0.13333	1	0	28 $(= y_2)$
(−5.00000)	0	1.00000	0	1	690 $(= z)$

Dependent variables: $\{x_2, y_2, z\}$
Independent variables: $x_1 = y_1 = 0$
Extreme point: $(x_1, x_2) = (0, 23)$
Value of objective function: $z = 690$

Next, we go back to **Step 2**.

Entering variable: x_1

x_1	x_2	y_1	y_2	z	RHS	Ratio
0.66667	1	0.03333	0	0	23	34.5 (= 23/0.66667)
2.33333	0	−0.13333	1	0	28	(12.0) (= 28/2.33333) ⟵ Exiting variable
(−5.00000)	0	1.00000	0	1	690	*

Step 2: optimality, choosing the entering variable (the variable with most negative coefficient x_1).

Step 3: feasibility, choosing the leaving variable (the variable with the smallest positive ratio y_2).

x_1	x_2	y_1	y_2	z	RHS	Ratio
0.66667	1	0.03333	0	0	23	34.5 (= 23/0.66667)
2.33333	0	−0.13333	1	0	28	(12.0) (= 28/2.33333) ⟵ Exiting variable
(−5.00000)	0	1.00000	0	1	690	*

Entering variable: x_1

Step 4 : pivot,

x_1	x_2	y_1	y_2	z	RHS
0	1	0.071429	−0.28571	0	15 $(= x_2)$
1	0	−0.057143	0.42857	0	12 $(= x_1)$
0	0	0.714286	2.14286	1	750 $(= z)$

Dependent variables: $\{x_2, x_1, z\}$
Independent variables: $y_1 = y_2 = 0$
Extreme point: $(x_1, x_2) = (12, 15)$
Value of objective function: $z = 750$

Next, we go back to **Step 2**.

x_1	x_2	y_1	y_2	z	RHS
0	1	0.071429	−0.28571	0	15 ($= x_2$)
1	0	−0.057143	0.42857	0	12 ($= x_1$)
0	0	0.714286	2.14286	1	750 ($= z$)

Dependent variables: $\{x_2, x_1, z\}$
Independent variables: $y_1 = y_2 = 0$
Extreme point: $(x_1, x_2) = (12, 15)$
Value of objective function: $z = 750$

Since no negative coefficients in the last row, we are done.

The optimal solution is $x_1 = 12, x_2 = 15$ and the value of the objective function is $z = 750$.

Another example

Solve

$$\text{maximize} \quad 3x_1 + x_2$$

subject to

$$2x_1 + x_2 \leq 6,$$
$$x_1 + 3x_2 \leq 9,$$

where x_1 and $x_2 \geq 0$.

As before, we can write the above as

$$2x_1 + x_2 + y_1 = 6,$$
$$x_1 + 3x_2 + y_2 = 9,$$
$$-3x_1 - x_2 + z = 0.$$

Next put these equations into a tableau format.

$$
\begin{aligned}
2x_1 + x_2 + y_1 &= 6, \\
x_1 + 3x_2 + y_2 &= 9, \\
-3x_1 - x_2 + z &= 0.
\end{aligned}
$$

Step 1 : initialize,

x_1	x_2	y_1	y_2	z	RHS
2	1	1	0	0	$6\,(=y_1)$
1	3	0	1	0	$9\,(=y_2)$
(−3)	−1	0	0	1	$0\,(=z)$

Dependent variables: $\{y_1, y_2, z\}$
Independent variables: $x_1 = x_2 = 0$
Extreme point: $(x_1, x_2) = (0, 0)$
Value of objective function: $z = 0$

x_1	x_2	y_1	y_2	z	RHS	Ratio
2	1	1	0	0	6	(3) (= 6/2) ← Exiting variable
1	3	0	1	0	9	9 (= 9/1)
(−3)	−1	0	0	1	0	*

↑ Entering variable

Step **2**: optimality, choosing the entering variable (the variable with most negative coefficient x_1).

Step **3**: feasibility: choosing the leaving variable (the variable with the smallest positive ratio y_1).

x_1	x_2	y_1	y_2	z	RHS	Ratio
2	1	1	0	0	6	(3) (= 6/2) ← Exiting variable
1	3	0	1	0	9	9 (= 9/1)
(−3)	−1	0	0	1	0	*

↑ Entering variable

Step 4 : pivot,

x_1	x_2	y_1	y_2	z	RHS
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$3\,(=x_1)$
0	$\frac{5}{2}$	$-\frac{1}{2}$	1	0	$6\,(=y_2)$
0	$\frac{1}{2}$	$\frac{3}{2}$	0	1	$9\,(=z)$

Dependent variables: $\{x_1, y_2, z\}$
Independent variables: $x_2 = y_1 = 0$
Extreme point: $(x_1, x_2) = (3, 0)$
Value of objective function: $z = 9$

Next we go back to **Step 2**.

x_1	x_2	y_1	y_2	z	RHS
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$3\,(=x_1)$
0	$\frac{5}{2}$	$-\frac{1}{2}$	1	0	$6\,(=y_2)$
0	$\frac{1}{2}$	$\frac{3}{2}$	0	1	$9\,(=z)$

Dependent variables: $\{x_1, y_2, z\}$
Independent variables: $x_2 = y_1 = 0$
Extreme point: $(x_1, x_2) = (3, 0)$
Value of objective function: $z = 9$

Since no more negative coefficients in last row, we are done.

The optimal solution is $x_1 = 3, x_2 = 0$ and the optimal value is $z = 9$.

You can solve LP via linprog—a solver provided by scipy—in python.

Sensitivity analysis

Motivation: the constants used to formulate the LP problem are only estimates, or they may change over time.

Aim: how sensitive the optimal solution is to changes in the constants used to formulate the LP.

We use carpenter's problem as an example. Recall

$$\text{maximize} \quad f(x_1, x_2) = 25x_1 + 30x_2$$

subject to

$$\begin{aligned} 20x_1 + 30x_2 &\le 690, \\ 5x_1 + 4x_2 &\le 120, \end{aligned}$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

Case 1

Aim: study sensitivity of the solution to changes in coefficients of the objective function.

Given that, the carpenter produces 12 tables and 15 bookcases.

Q: is this still optimal if the net profit of table is changed?

That is, we consider the following

$$\text{maximize} \quad f(x_1, x_2) = cx_1 + 30x_2$$

subject to

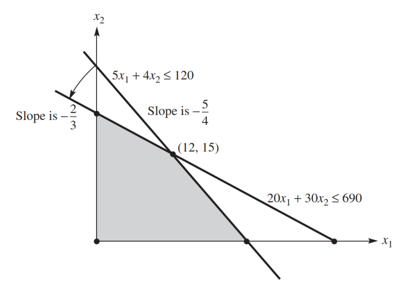
$$20x_1 + 30x_2 \leq 690,$$
$$5x_1 + 4x_2 \leq 120,$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

For what values of c the solution $x_1 = 12, x_2 = 15$ is optimal?

Let z be the optimal value of $f(x_1, x_2) = cx_1 + 30x_2$.

The line $cx_1 + 30x_2 = z$ passes through $(12, 15)$ and intersects the feasible region only at the boundary.



This line has the slope $-c/30$.

$$-\frac{5}{4} \leq -\frac{c}{30} \leq -\frac{2}{3}$$
$$\Rightarrow 20 \leq c \leq 37.5.$$

Case 2

Aim: study the effect on the objective value if the resources are changed.

Consider the same example:

$$\text{maximize} \quad f(x_1, x_2) = 25x_1 + 30x_2$$

subject to

$$20x_1 + 30x_2 \leq 690,$$
$$5x_1 + 4x_2 \leq 120,$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

What happens if the available labor is increased by 1 unit?

That is, the second constraint becomes

$$5x_1 + 4x_2 \leq 121.$$

Hence, the problem is

$$\text{maximize} \quad f(x_1, x_2) = 25x_1 + 30x_2$$

subject to

$$20x_1 + 30x_2 \leq 690,$$
$$5x_1 + 4x_2 \leq 121,$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

Skipping the calculations, the optimal solution is

$$x_1 = 12.429, \quad x_2 = 14.714, \quad f = 752.14.$$

Therefore, the profit is increased by 2.14 units.

If one unit of labor costs less than 2.14 units, then it would be profitable to do so.

Integer linear programming

For some problems, the solutions should be integers.

We will introduce the Branch-and-Bound (BB) algorithm.

The idea is very simple:

- First solve the LP problem without the integer restrictions.
- Add additional constraints for each non-integer solution, and solve the LP problem again without the integer restriction.
- The geometric, algebraic or simplex method can be applied for solving LP problems.

We illustrate the idea by an example.

Example using BB algorithm

Consider the integer linear programming problem:

$$\max\ 5x_1 + 4x_2$$

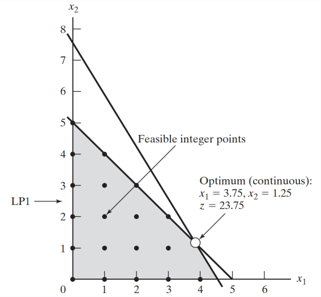
subject to

$$x_1 + x_2 \leq 5, \qquad 10x_1 + 6x_2 \leq 45,$$

$$x_1, x_2 \geq 0, \qquad x_1, x_2 \text{ are integers.}$$

The shaded area is the region defined by the inequalities.

The dots are feasible solutions.



Solve the LP problem without integer restrictions (LP1)

$$(\text{LP1}) \quad \max\ 5x_1 + 4x_2$$

subject to

$$x_1 + x_2 \le 5, \qquad 10x_1 + 6x_2 \le 45,$$

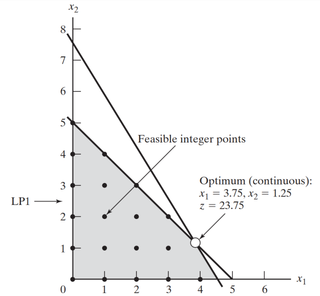
$$x_1, x_2 \ge 0.$$

By the geometric method, it is easy to see that

$$x_1 = 3.75, \quad x_2 = 1.25$$

and the objective function value is

$$z = 23.75.$$



From above, we see that both x_1 and x_2 are not integers.

By the BB algorithm, we choose one of them, and add constraints.

For example, we choose x_1.

Since $3 < x_1 < 4$, it does not contain any integer solution, and thus it can be removed from the feasible region of LP1 without affecting the original problem.

So, we introduce two new LP problems:

$$
\begin{aligned}
(\text{LP2}) \quad &= \quad (\text{LP1}) + (x_1 \leq 3), \\
(\text{LP3}) \quad &= \quad (\text{LP1}) + (x_1 \geq 4).
\end{aligned}
$$

$$(\text{LP2}) = (\text{LP1}) + (x_1 \leq 3),$$
$$(\text{LP3}) = (\text{LP1}) + (x_1 \geq 4).$$

The solutions can be found easily:

- For LP2

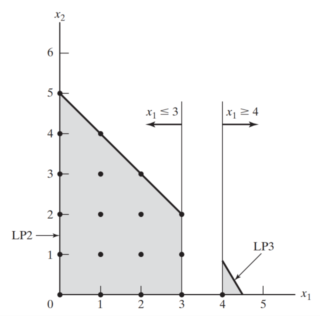
$$x_1 = 3, x_2 = 2,$$

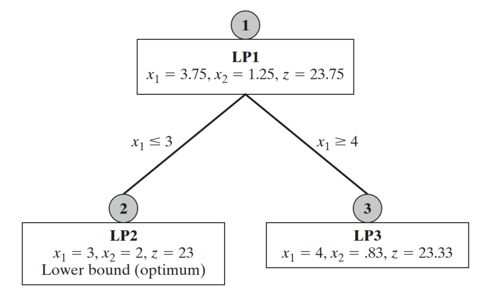
$$z = 23.$$

- For LP3

$$x_1 = 4, x_2 = 0.83,$$

$$z = 23.33.$$





- LP2 has an integer solution, no further action is needed;
- LP3 does not have an integer solution, we need to branch again, and remove the region $0 < x_2 < 1$.

$$(\text{LP3}) \quad = \quad (\text{LP1}) + (x_1 \geq 4)$$

We introduce two new problems

$$(\text{LP4}) \quad = \quad (\text{LP3}) + (x_2 \leq 0) = (\text{LP1}) + (x_1 \geq 4) + (x_2 \leq 0),$$

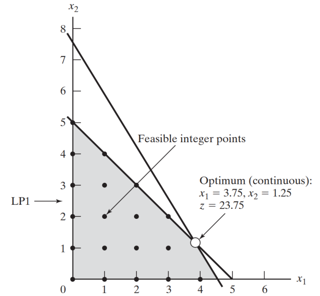
$$(\text{LP5}) \quad = \quad (\text{LP3}) + (x_2 \geq 1) = (\text{LP1}) + (x_1 \geq 4) + (x_2 \geq 1).$$

- For LP4

$$x_1 = 4.5, x_2 = 0,$$

$$z = 22.5,$$

  branch needed.
- For LP5, no feasible solution.



$$(\text{LP4}) \quad = (\text{LP1}) + (x_1 \geq 4) + (x_2 \leq 0)$$

From above, we remove the region $4 < x_1 < 5$, and introduce two new LPs:

$$(\text{LP6}) = (\text{LP4}) + (x_1 \leq 4) = (\text{LP1}) + (x_1 \geq 4) + (x_2 \leq 0) + (x_1 \leq 4)$$

$$(\text{LP7}) = (\text{LP4}) + (x_1 \geq 5) = (\text{LP1}) + (x_1 \geq 4) + (x_2 \geq 1) + (x_1 \geq 5)$$

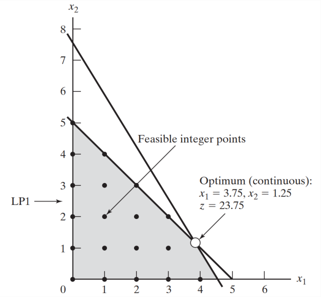
- For LP6

$$x_1 = 4, x_2 = 0,$$

$$z = 20,$$

  integer solution.

- For LP7, no feasible solution.

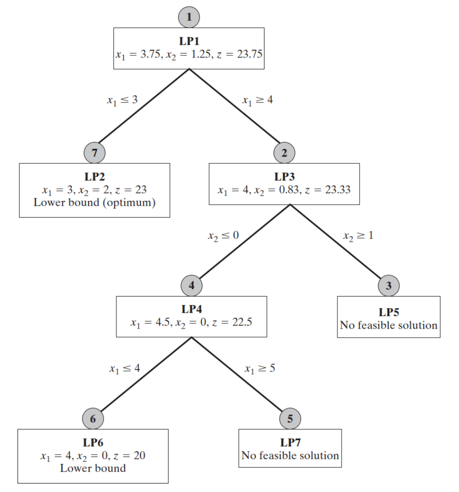


Compare all cases with integer solutions, namely, LP2 and LP6, we get the solution:

$$x_1 = 3, x_2 = 2,$$

$$z = 23.$$

The python library scipy also provides a solver called milp for integer linear programming.



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