MATH 3290 Mathematical Modeling

Chapter 12: Modeling with Systems of Differential Equations

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Course webpage

https://www.math.cuhk.edu.hk/course/2324/math3290

SCAN ME

We will discuss modeling with a system of differential equations.

Here, a system can model interactions among variables.

Note: Since analytical solutions cannot be found easily, we will discuss the qualitative behaviors of the solution by the graphical method. We will also introduce a numerical approximation method.

Graphical solutions

Consider the following system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = g(x, y).$$

The solutions are $x(t)$ and $y(t)$.

We interpret the solution is the position $(x(t), y(t))$ of a particle at time t. The xy-plane is called the phase plane.

As t varies, $(x(t), y(t))$ defines a path (or trajectory or orbit) in the phase plane.

The particle moves in the phase plane in the direction of increasing t.

Recall

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = g(x,y).$$

An equilibrium point (EP) (x_0, y_0) is a point for which $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} = 0$. That is,

$$f(x_0, y_0) = 0, \qquad g(x_0, y_0) = 0.$$

Stability of equilibrium point (EP): we say (x_0, y_0) is

- stable if any path starts close to the point remains close for all future time;
- asymptotically stable if it is stable and the path approaches to the point as t tends to infinity;
- unstable if it is not stable.

An example: Consider

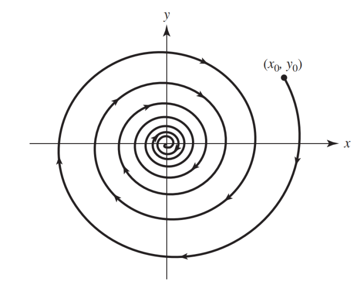
$$\frac{dx}{dt} = -x + y, \qquad \frac{dy}{dt} = -x - y.$$

It is easy to check that a solution

$$x(t) = e^{-t} \sin t, \qquad y(t) = e^{-t} \cos t.$$

The illustration shows that

- a path with the initial position (x_0, y_0);
- the particle moves in the direction of increasing t;
- $(0, 0)$ is an asymptotically stable equilibrium point (EP).

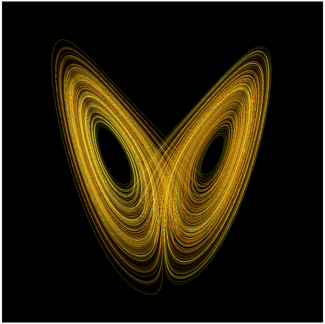


Lorenz system

The Lorenz system:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \sigma(y - x)\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x(\rho - z) - y\\ \frac{\mathrm{d}z}{\mathrm{d}t} &= xy - \beta z.\end{aligned}$$

Note σ, ρ and β are parameters.

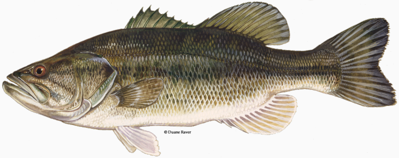


A trajectory of the Lorenz system.

- If $\rho < 1$, there exists one and only one asymptotically stable equilibrium point.
- If $\rho = 28$, $\sigma = 10$, and $\beta = 8/3$, the Lorenz system has chaotic solutions.

Suppose there are two types of fish—trout and bass.

Trout



Bass

A competitive hunter model

Suppose there are two types of fish—trout and bass.

We build a model to describe the interaction of them. We assume that they compete for some limited resources, say food.

Let $x(t)$ and $y(t)$ be the populations of trout and bass, respectively.

Assumption 1: without the existence of bass, trout will grow without limit, so we propose the following model

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax, \qquad a > 0.$$

It says that the rate of change of trout population is proportional to its population.

Assumption 2: when bass exists, they will limit the growth of trout because the two species will compete for food.

We model the decrease in the population by the product of x and y, so we propose the following model

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bxy, \qquad b > 0.$$

Following the same reasoning, we propose the following model for the rate of change of bass population

$$\frac{\mathrm{d}y}{\mathrm{d}t} = my - nxy, \qquad m, n > 0.$$

Graphical analysis

The model is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(a - by), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = (m - nx)y.$$

We will look at the phase plane.

Step 1: locate the equilibrium points (EPs),

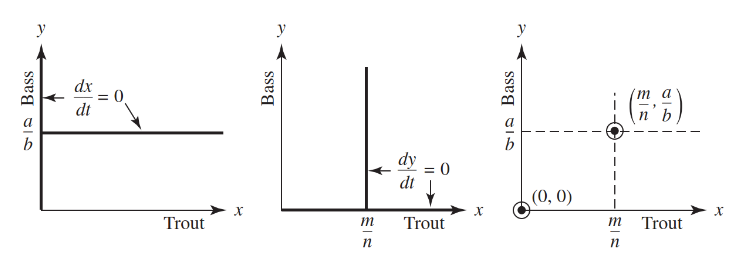
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} = 0, \quad \Rightarrow \quad x(a - by) = 0,\ (m - nx)y = 0.$$

Thus, there are 2 equilibrium points (EPs): $(0, 0)$ and $(m/n, a/b)$.

Step 2: draw the lines where $\mathrm{d}x/\mathrm{d}t = 0$ or $\mathrm{d}y/\mathrm{d}t = 0$.

Note: $\mathrm{d}x/\mathrm{d}t = 0$ when $x = 0$ or $y = a/b$, and $\mathrm{d}y/\mathrm{d}t = 0$ when $y = 0$ or $x = m/n$.

The above information are shown in the following figures.

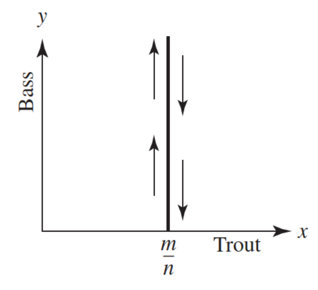


Recall:

$\mathrm{d}x/\mathrm{d}t = 0$ when $x = 0$ or $y = a/b$, and $\mathrm{d}y/\mathrm{d}t = 0$ when $y = 0$ or $x = m/n$. The lines divide the phase plane into 4 regions.

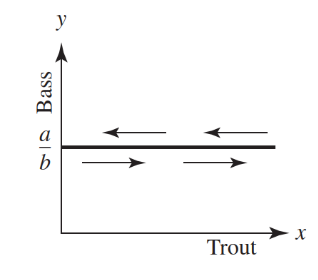
Step 3: determine movement of the particle in each region. First, look at the lines where $\mathrm{d}x/\mathrm{d}t = 0$ or $\mathrm{d}y/\mathrm{d}t = 0$ again.

- The line $x = m/n$ is shown.
- On the left, $x < m/n$, so $\mathrm{d}y/\mathrm{d}t = (m - nx)y > 0$, thus the particle always moves up.
- On the right, $x > m/n$, so $\mathrm{d}y/\mathrm{d}t = (m - nx)y < 0$, thus the particle always moves down.



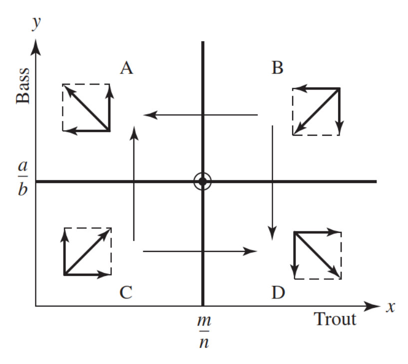
The line of $\mathrm{d}y/\mathrm{d}t = 0$

- The line $y = a/b$ is shown.
- In the lower region, $y < a/b$, so $\mathrm{d}x/\mathrm{d}t = x(a - by) > 0$, thus the particle always moves to the right.
- In the upper region, $y > a/b$, so $\mathrm{d}x/\mathrm{d}t = x(a - by) < 0$, thus the particle always moves to the left.



The line of $\mathrm{d}x/\mathrm{d}t = 0$

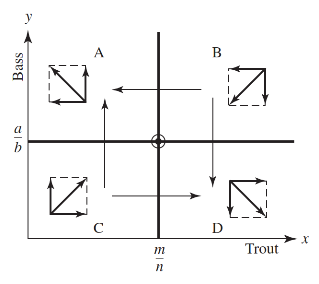
Combining the above analysis, we obtain the following figure.



Step 4: determine stability of equilibrium points (EPs).

Consider the point $(0, 0)$:

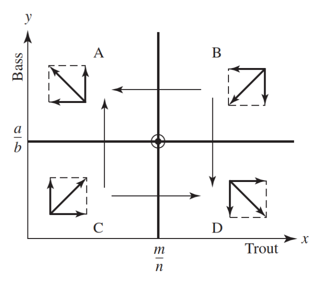
- if the particle starts near $(0, 0)$, which is in region C,
- clearly, the particle will move away from $(0, 0)$.
- $(0, 0)$ is unstable.



Stability of the other equilibrium point (EP).

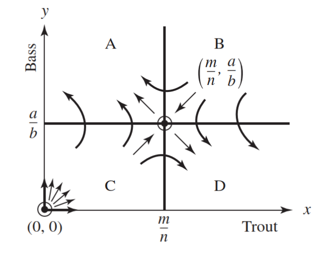
Consider the point $(m/n, a/b)$:

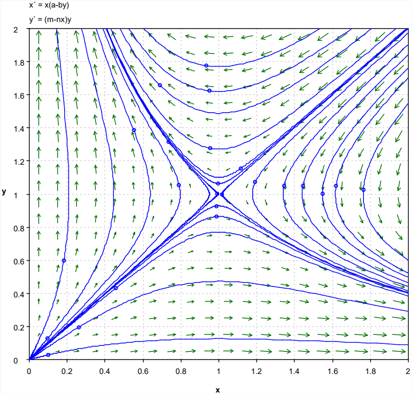
- if the particle starts near $(m/n, a/b)$, and in region D,
- clearly, the particle will move away from $(m/n, a/b)$.
- $(m/n, a/b)$ is unstable.



Step **5**: model interpretation.

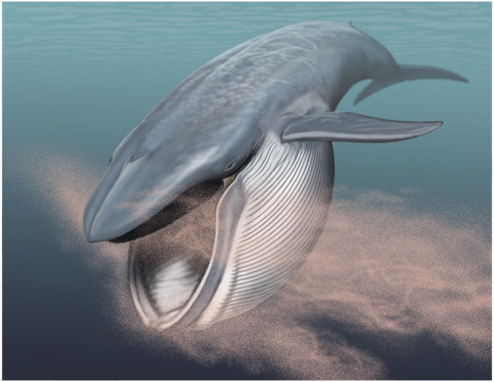
- $(m/n, a/b)$ is unstable, thus co-existence is impossible.
- the initial condition is crucial to the outcome:
  - if starts in region A, bass dominates;
  - if starts in region D, trout dominates;
  - if starts in regions B or C, either can happen.





Program for 2D phase plots: pplane. You can download it from `https://www.cs.unm.edu/~joel/dfield/` (You need Java Runtime Environment to run it).

Suppose there are two types of species—whale and krill.



Whale and krill

A predator-prey model

Suppose there are two types of species—whale and krill.

We build a model to describe the interaction of them. We assume that whales eat the krill.

Let $x(t)$ and $y(t)$ be the populations of krill and whales, respectively.

Assumption 1: without the existence of whales, krill will grow without limit, so we propose the following model

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax, \qquad a > 0.$$

It says that the rate of change of krill population is proportional to its population.

Assumption 2: when whales exist, they will limit the growth of krill because whales will eat krill.

We model the decrease in the population by the product of x and y, so we propose the following model

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bxy, \qquad b > 0.$$

That is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(a - by).$$

Assumption 3: without the existence of krill, the population of whales will decline, so we propose the following model

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -my, \qquad m > 0.$$

It says that the rate of decay of the whale population is proportional to its population.

Assumption 4: when krill exist, they will provide foods to whales, and this will increase the whale population.

We model the increase in the population by the product of x and y, so we propose the following model

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -my + nxy, \qquad n > 0.$$

Graphical analysis

The model is

$$\frac{dx}{dt} = x(a - by), \qquad \frac{dy}{dt} = (-m + nx)y.$$

We will look at the phase plane.

Step 1: locate the equilibrium points (EPs),

$$\frac{dx}{dt} = \frac{dy}{dt} = 0, \quad \Rightarrow \quad x(a - by) = 0, \; (-m + nx)y = 0.$$

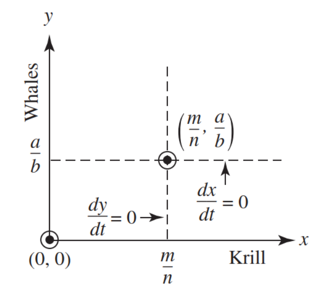
Thus, there are 2 equilibrium points (EPs): $(0, 0)$ and $(m/n, a/b)$.

Step 2: draw the lines where $dx/dt = 0$ or $dy/dt = 0$.

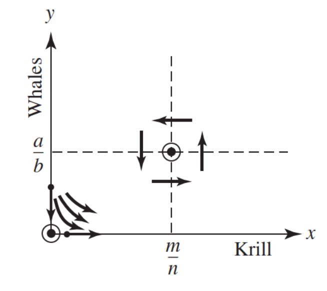
Note: $dx/dt = 0$ when $x = 0$ or $y = a/b$, and $dy/dt = 0$ when $y = 0$ or $x = m/n$. These lines divide the phase plane into four regions.

Step 3: determine movement of the particle in each region.

- On the left, $x < m/n$, so $\mathrm{d}y/\mathrm{d}t < 0$, and the particle moves down.
- On the right, $x > m/n$, so $\mathrm{d}y/\mathrm{d}t > 0$, and the particle moves up.
- In the lower region, $y < a/b$, so $\mathrm{d}x/\mathrm{d}t > 0$, particle moves to right.
- In the upper region, $y > a/b$, so $\mathrm{d}x/\mathrm{d}t < 0$, particle moves to left.



Hence, we obtain the following phase plane.



Step 4: determine stability of equilibrium points (EPs),

From above, it is clear that $(0, 0)$ is unstable.

The stability of $(m/n, a/b)$ is not clear. Looks like the phase lines rotate anticlockwise around it.

We present further mathematical analysis for $(m/n, a/b)$.

Recall that the model is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(a - by), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = (-m + nx)y.$$

We find a relation of x and y (i.e., a curve in the phase plane).

By the chain rule and the inverse function theorem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t}.$$

Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(-m + nx)y}{x(a - by)}.$$

We separate the variables,

$$\left(\frac{a}{y} - b\right)\,\mathrm{d}y = \left(n - \frac{m}{x}\right)\,\mathrm{d}x.$$

Integrate both sides

$$\int \left(\frac{a}{y} - b\right)\,\mathrm{d}y = \int \left(n - \frac{m}{x}\right)\,\mathrm{d}x.$$

So,

$$a \ln y - by = nx - m \ln x + k_1, \qquad k_1 \text{ is a constant.}$$

Finally, we have

$$\frac{y^a}{e^{by}} = K\frac{e^{nx}}{x^m}, \qquad K \text{ is a constant.}$$

Recall

$$\frac{y^a}{e^{by}} = K\frac{e^{nx}}{x^m}.$$

Let $f(y) = y^a e^{-by}$ and $g(x) = x^m e^{-nx}$. Then we have

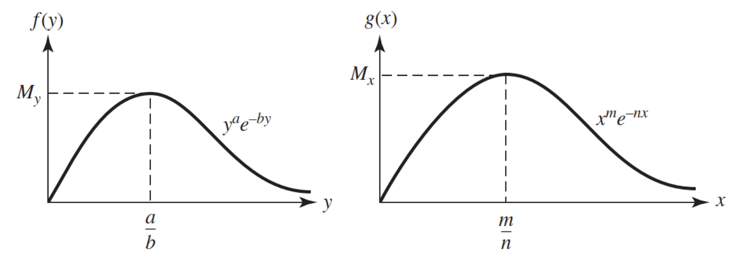
$$f(y)\,g(x) = K.$$

Note this K should be determined by the initial condition $(x(0), y(0))$, different K implies different phase lines.

We first state some properties of $f(y)$ and $g(x)$:

- $f(0) = 0$ and $g(0) = 0$;
- f and g tends to zero as y and x tends to infinity;
- f has a local(global) maximum at $y = a/b$, g has a local(global) maximum at $x = m/n$.

We have the following sketch for $f(y)$ and $g(x)$



Here, M_y is the maximum value of $f(y)$, and M_x is the maximum value of $g(x)$.

Now, we look at the equation $f(y)\,g(x) = K$.

We consider three cases: $K > M_y M_x$, $K = M_y M_x$ and $K < M_y M_x$.

Case 1: $K > M_y M_x$.

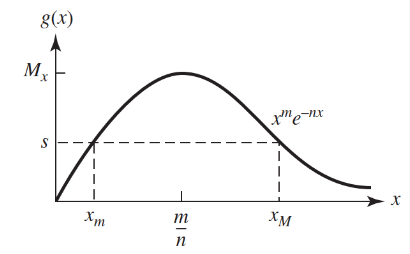
Clearly, the equation $f(y)\,g(x) = K$ has no solution.

Case 2: $K = M_y M_x$.

Clearly, the equation $f(y)\,g(x) = K$ has exactly one solution, which is $x = m/n$ and $y = a/b$. This is just the equilibrium point $(m/n, a/b)$.

Case 3: $K < M_y M_x$.

We write $K = sM_y$ and $s < M_x$. The equation $g(x) = s$ has two solutions, $x = x_m$ and $x = x_M$.



Recall, we are looking at the solution of $f(y)\,g(x) = K$.

Case 3a: if $x < x_m$ or $x > x_M$, we have $g(x) < s$ and

$$f(y) = K/g(x) = (sM_y)/g(x) > M_y, \quad \text{since } g(x) < s.$$

Hence, no solution.

Case 3b: if $x = x_m$ or $x = x_M$, we have $g(x) = s$ and

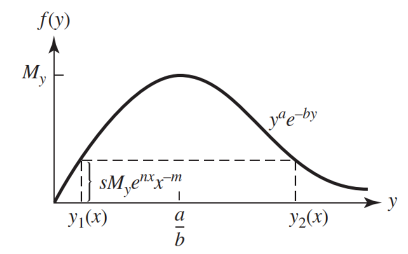
$$f(y) = K/g(x) = (sM_y)/s = M_y.$$

Hence, two solutions $(x_m, a/b)$ and $(x_M, a/b)$.

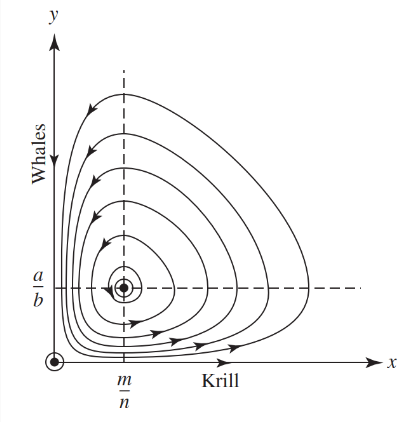
Case 3c: if $x_m < x < x_M$, we have $g(x) > s$ and

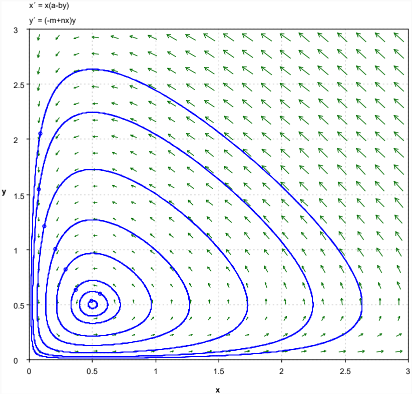
$$f(y) = K/g(x) = (sM_y)/g(x) < M_y, \quad \text{since } g(x) > s.$$

Thus, we are able to find two solutions $(x, y_1(x))$ and $(x, y_2(x))$, where $x_m < x < x_M$.



Combining all the above discussions, we see that the trajectories are periodic near $(m/n, a/b)$.

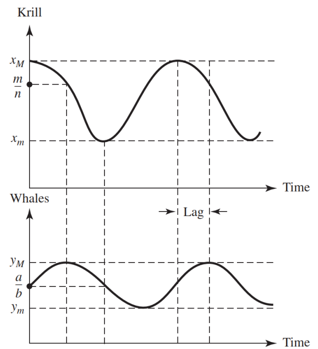




Phase lines from pplane

Step **5**: model interpretation.

- Co-existence of whales and krill are possible, the point $(m/n, a/b)$ is stable.
- If starts at a point in $x < m/n$ and $y > a/b$ (EP), both populations will decrease.
- Similar for the other three cases.
- The two populations fluctuate between their maximum and minimum values.



Effects of harvesting

Recall the model for whales and krill population is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(a - by), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = (-m + nx)y.$$

Let T be the time of one complete cycle.

We define the average levels over the cycle by

$$\bar{x} = \frac{1}{T}\int_0^T x(t)\ \mathrm{d}t, \qquad \bar{y} = \frac{1}{T}\int_0^T y(t)\ \mathrm{d}t.$$

We should have $x(0) = x(T)$ and $y(0) = y(T)$.

From the first differential equation

$$\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t} = a - by.$$

Integrating with respect to t, then

$$\int_0^T \frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t = \int_0^T (a - by)\ \mathrm{d}t$$
$$\Rightarrow \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}(\ln x(t))\ \mathrm{d}t = aT - bT\bar{y}$$
$$\Rightarrow \ln x(T) - \ln x(0) = aT - bT\bar{y}.$$

Since $x(T) = x(0)$, we have

$$\bar{y} = \frac{a}{b}.$$

By the similar techniques, we have

$$\bar{x} = \frac{m}{n}.$$

Hence, the average levels are exact the equilibrium points.

We assume that the fishing of krill will decrease its population at a rate $rx(t)$.

Since there is less food for whales, its population will also decrease at a rate $ry(t)$.

We have the new model

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x\left((a - r) - by\right), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \left(-(m + r) + nx\right)y.$$

Using the same steps, the new average levels are

$$\bar{x} = \frac{m + r}{n}, \qquad \bar{y} = \frac{a - r}{b}.$$

We see that, fishing of krill will actually increase the average level of krill, and decrease the average level of whales.

This is known as Volterra's principle.

An arms race

Consider two countries engaged in an arms race.

Let $x(t)$ be annual defense expenditure for Country 1 and $y(t)$ be annual defense expenditure for Country 2.

Assumption 1: the expenditure decreases at a rate proportional to the current expenditure

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ax, \qquad a > 0.$$

Assumption 2: the increase in expenditure is proportional to the amount spend by the other country

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ax + by, \qquad b > 0.$$

Assumption 3: even if the defense expenditure for both countries are zero, Country 1 still needs to increase its defense expenditure because of possible future action of Country 2

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ax + by + c, \qquad c > 0.$$

By a similar argument, we propose the following model

$$\frac{\mathrm{d}y}{\mathrm{d}t} = mx - ny + p,$$

where m, n and $p > 0$.

Recall that the model is

$$\frac{dx}{dt} = -ax + by + c, \qquad \frac{dy}{dt} = mx - ny + p.$$

Step 1: locate the equilibrium points (EPs),

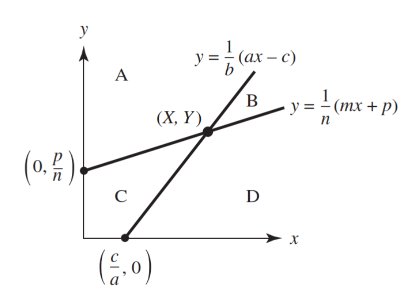
$$\frac{dx}{dt} = \frac{dy}{dt} = 0, \quad \Rightarrow \quad -ax + by + c = 0,\ mx - ny + p = 0.$$

Solving, the only equilibrium point (EP) (X, Y) is

$$X = \frac{bp + cn}{an - bm}, \qquad Y = \frac{ap + cm}{an - bm}.$$

We need to assume that $an - bm > 0$ so that $X, Y > 0$.

Step 2: draw the lines where $\mathrm{d}x/\mathrm{d}t = 0$ or $\mathrm{d}y/\mathrm{d}t = 0$. The two lines are $y = \frac{1}{b}(ax - c)$ and $y = \frac{1}{n}(mx + p)$.



The two lines divide the phase plane into 4 regions.

(Note that $a/b > m/n$ by our assumption.)

Step 3: determine movement of the particle in each region.

- Region A:

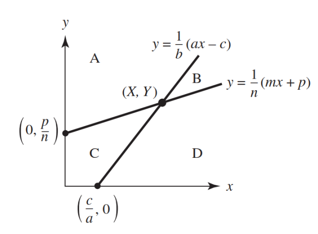
$$y > \frac{1}{b}(ax - c),\ y > \frac{1}{n}(mx + p),$$

so, $\mathrm{d}x/\,\mathrm{d}t > 0$ and $\mathrm{d}y/\,\mathrm{d}t < 0$.

- Region B:

$$y < \frac{1}{b}(ax - c),\ y > \frac{1}{n}(mx + p),$$

so, $\mathrm{d}x/\,\mathrm{d}t < 0$ and $\mathrm{d}y/\,\mathrm{d}t < 0$.



- Region C:

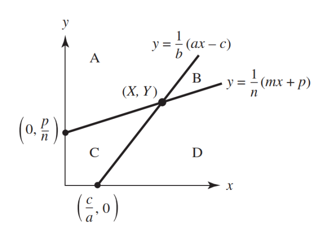
$$y > \frac{1}{b}(ax - c),\ y < \frac{1}{n}(mx + p),$$

so, $\mathrm{d}x/\mathrm{d}t > 0$ and $\mathrm{d}y/\mathrm{d}t > 0$.

- Region D:

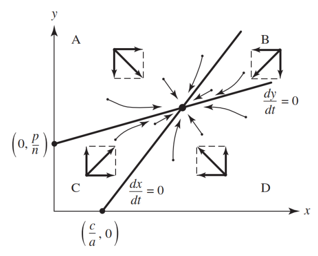
$$y < \frac{1}{b}(ax - c),\ y < \frac{1}{n}(mx + p),$$

so, $\mathrm{d}x/\mathrm{d}t < 0$ and $\mathrm{d}y/\mathrm{d}t > 0$.



Combining the above.

- A: $dx/dt > 0$ and $dy/dt < 0$.
- B: $dx/dt < 0$ and $dy/dt < 0$.
- C: $dx/dt > 0$ and $dy/dt > 0$.
- D: $dx/dt < 0$ and $dy/dt > 0$.



Steps 4 and 5: stability of equilibrium points (EPs), model interpretation.

We see that (X, Y) is asymptotically stable. In the long run, the expenditure for Countries 1 and 2 are X and Y.

Euler's method

Consider the system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x, y) \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = g(t, x, y)$$

with initial conditions

$$x(t_0) = x_0, \qquad y(t_0) = y_0.$$

We use the Euler's method to find an approximate solution for $t \geq t_0$.

Idea: similar to the case with one differential equation, we approximate the solution values by the values of tangent lines.

The tangent line at the point (t_0, x_0) is

$$T(t) = x_0 + \frac{\mathrm{d}x}{\mathrm{d}t}(t_0)(t - t_0).$$

By the system, we have

$$T(t) = x_0 + f(t_0, x_0, y_0)(t - t_0).$$

Let $t_1 = t_0 + \Delta t$. Then we can use the value $T(t_1)$:

$$x_1 = x_0 + f(t_0, x_0, y_0)\Delta t$$

as an approximation of $x(t_1)$.

Similarly, the tangent line at the point (t_0, y_0) is

$$S(t) = y_0 + \frac{\mathrm{d}y}{\mathrm{d}t}(t_0)(t - t_0).$$

By the system, we have

$$S(t) = y_0 + g(t_0, x_0, y_0)(t - t_0).$$

Let $t_1 = t_0 + \Delta t$. Then we can use the value $S(t_1)$:

$$y_1 = y_0 + g(t_0, x_0, y_0)\Delta t$$

as an approximation of $y(t_1)$.

Combining the above calculations,

$$x_1 = x_0 + f(t_0, x_0, y_0)\Delta t,$$
$$y_1 = y_0 + g(t_0, x_0, y_0)\Delta t.$$

In general, we let

$$t_n = t_0 + n\Delta t$$

and let

$$\begin{aligned} x_n &= \text{ approximation of } x(t_n), \\ y_n &= \text{ approximation of } y(t_n). \end{aligned}$$

The above shows that we can find x_n, y_n by

Euler's method

$$\begin{aligned} x_{n+1} &= x_n + f(t_n, x_n, y_n)\Delta t, \\ y_{n+1} &= y_n + g(t_n, x_n, y_n)\Delta t. \end{aligned}$$

Example: competitive hunter model (refined)

Suppose there are two types of fish: trout and bass.

We build a model to describe the interaction of them. We assume that they compete for some limited resources, say food.

Let $x(t)$ and $y(t)$ be the population of trout and bass, respectively.

Assumption 1: without the existence of bass, trout will grow with limit, so we propose the following model

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax(M - x), \qquad a, M > 0.$$

Assumption 2: when bass exists, they will limit the growth of trout because the two species will compete for food.

We model the decrease in the population by the product of x and y, so we propose the following model

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax(M-x) - bxy, \qquad b > 0.$$

Following the same reasoning, we propose the following model for the rate of change of bass population

$$\frac{\mathrm{d}y}{\mathrm{d}t} = my(N-y) - nxy, \qquad m, n, N > 0.$$

Specifically, we consider

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(10 - x - y),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(15 - x - 3y).$$

Suppose that, initially, $x(0) = 5$ and $y(0) = 2$.

We use the Euler's method to predict the long term behavior.

We will compute the solution for $0 \le t \le 7$ with $\Delta t = 0.1$. So, we need to perform 70 iterations.

Step 0: $x_0 = 5$ and $y_0 = 2$.

Step 1:

$$x_1 = x_0 + f(t_0, x_0, y_0)\Delta t = 5 + 0.1x_0(10 - x_0 - y_0) = 6.5,$$
$$y_1 = y_0 + g(t_0, x_0, y_0)\Delta t = 2 + 0.1y_0(15 - x_0 - 3y_0) = 2.8.$$

Note that x_1, y_1 are approximate values of $x(0.1)$ and $y(0.1)$.

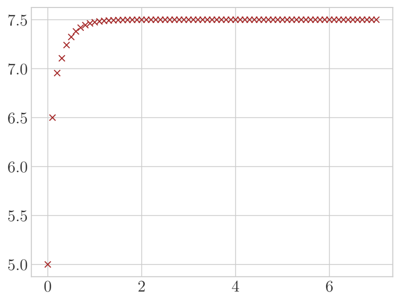
Step 2:

$$x_2 = x_1 + f(t_1, x_1, y_1)\Delta t = 6.5 + 0.1x_1(10 - x_1 - y_1) = 6.955,$$
$$y_2 = y_1 + g(t_1, x_1, y_1)\Delta t = 2.8 + 0.1y_1(15 - x_1 - 3y_1) = 2.828.$$

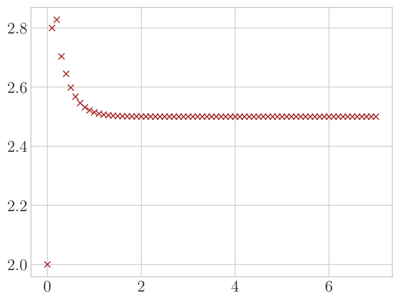
Note that x_2, y_2 are approximate values of $x(0.2)$ and $y(0.2)$.

Continue until Step 70.

We can plot the approximate values against time:

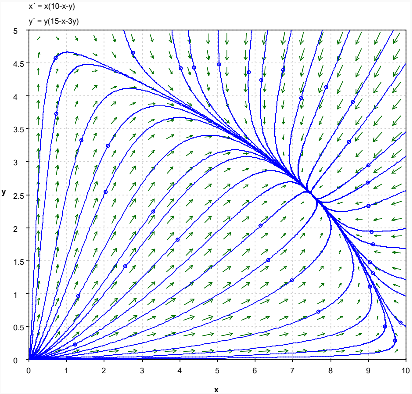


Plot of $x(t)$



Plot of $y(t)$

We see that the solutions converge to the equilibrium value $(7.5, 2.5)$.



Phase lines from pplane

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