MATH 3290 Mathematical Modeling

Chapter 1: Modeling Change

Kuang HUANG

January 16, 2024

Department of Mathematics
The Chinese University of Hong Kong

Course webpage

https://www.math.cuhk.edu.hk/course/2324/math3290

SCAN ME

Modeling change

A powerful paradigm for modeling change is:

$$\texttt{future\_value} = \texttt{present\_value} + \texttt{change}.$$

To predict the `future_value`, one needs to know the `present_value` and the `change`.

Note, `change` is something that we need to determine.

Thus, we need to develop mathematical models that give predictions to `change`.

Modeling change

We need a mathematical model for change. Note that

$$\texttt{change} = \texttt{future\_value} - \texttt{present\_value}.$$

- If the behavior is taking place over discrete time periods, we will use a model based on difference equation, which will be studied in this chapter.
- If the behavior is taking place continuously with respect to time, we will use a model based on differential equation, which will be studied later in this course.

Modeling with difference equations

Some notations:

Let $a_0, a_1, a_2, \ldots$ be a sequence. They represent the values of certain variable at discrete times $0, 1, 2, \ldots$

The changes are defined by

$$\begin{aligned} \Delta a_0 &= a_1 - a_0, \\ \Delta a_1 &= a_2 - a_1, \\ &\vdots \end{aligned}$$

In general, the change at time n is $\Delta a_n = a_{n+1} - a_n$.

Example: a saving account

Consider a saving account with an initial deposit of \$1000. Assume that the interest rate is 1% per month.

Let n be the number of months and a_n be the amount at the end of the n-th month.

The change at the n-th month is

$$\Delta a_n = 0.01 a_n.$$

Thus, we obtain the following difference equation:

$$a_{n+1} - a_n = 0.01 a_n$$

or

$$a_{n+1} = 1.01 a_n.$$

Since the initial deposit is \$1000, we set $a_0 = 1000$.

We obtain the following discrete dynamical system model

$$
\begin{aligned}
a_{n+1} &= 1.01a_n \qquad n = 0, 1, 2, \dots, \\
a_0 &= 1000.
\end{aligned}
$$

Remarks:

- In this example, the change Δa_n is a function of a_n. In general, the change Δa_n can be a function of more terms in the sequence, that is, $\Delta a_n = f(a_n, a_{n-1}, \dots)$.
- The change Δa_n can also be a function of some other external quantities.

Example: home mortgage

Consider a home loan of \$80,000. The monthly interest rate is 1% and the monthly payment is \$880.87.

You want to know how much you owe at the end of 72 months.

We construct a model for this problem. Let b_n be the amount owe at the end of the n-th month. Note

$$\begin{aligned}\texttt{change\_in\_amount\_owe} =& \texttt{interest\_incurred}\\ &- \texttt{monthly\_payment}.\end{aligned}$$

So,

$$\Delta b_n = 0.01 b_n - 880.87.$$

That is

$$b_{n+1} - b_n = 0.01 b_n - 880.87.$$

Thus we obtain the model

$$\begin{aligned} b_{n+1} &= 1.01 b_n - 880.87 \qquad n = 0, 1, 2, \ldots, \\ b_0 &= 80{,}000. \end{aligned}$$

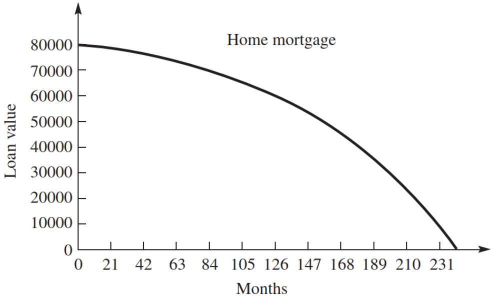
One can find the numerical solution in the following way.

$$\begin{aligned} b_1 &= 1.01 b_0 - 880.87 = 1.01 * 80{,}000 - 880.87 = 79{,}919.13, \\ b_2 &= 1.01 b_1 - 880.87 = 1.01 * 79{,}919.13 - 880.87 = 79{,}837.45, \\ &\vdots \end{aligned}$$

Continuing, we get $b_{72} = 71532.11$.

The numerical solution of the model.

Months n	Amount owed b_n
0	80000.00
1	79919.13
2	79837.45
3	79754.96
4	79671.64
5	79587.48
6	79502.49
7	79416.64
8	79329.94
9	79242.37
10	79153.92
11	79064.59
12	78974.37



One can find the number of months needed to pay off the loan.

Approximating change

In most cases, the change will not be as precise a procedure as in the examples of saving account and home mortgage.

Typically, we need to obtain some data, plot the change and observe a pattern. And then approximate the change in mathematical terms.

That is, we usually first do something like

$$\texttt{Plot}(\texttt{a\_certain\_variable}, \texttt{change}, \ldots).$$

Mathematically, a pattern can be represented by a function

$$\texttt{change} = \texttt{Function}(\texttt{a\_certain\_variable}).$$

Example: growth of yeast

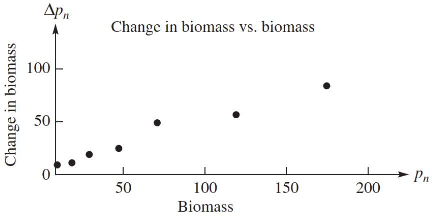
Time in hours n	Observed yeast biomass p_n
0	9.6
1	18.3
2	29.0
3	47.2
4	71.1
5	119.1
6	174.6
7	257.3

- Experiments contain measurements of yeast biomass at different time points.
- Predict future biomass or explain the way biomass changes.

To find a model, we look at the change.

We plot the change against p_n.

Time in hours n	Observed yeast biomass p_n	Change in biomass $p_{n+1} - p_n$
0	9.6	8.7
1	18.3	10.7
2	29.0	18.2
3	47.2	23.9
4	71.1	48.0
5	119.1	55.5
6	174.6	82.7
7	257.3	



From the graph, it is reasonable to assume that the change Δp_n is proportional to p_n. That is

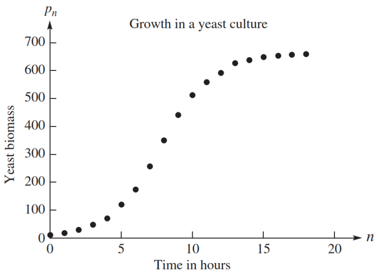
$$\Delta p_n = k\, p_n \qquad k > 0.$$

We obtain the model for yeast biomass (or population).

$$p_{n+1} = (1+k)p_n$$

This model predicts that yeast population will increase forever.

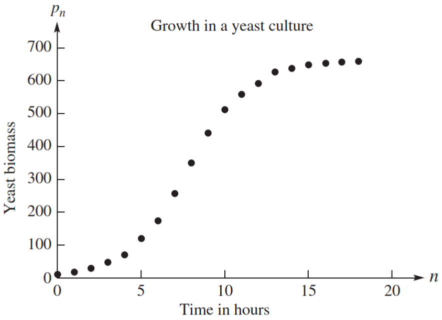
It shows that the actual population does not increase forever. We need to refine our model.



Remark

Model refinement is an important step.

Time in hours n	Yeast biomass p_n	Change/hour $p_{n+1} - p_n$
0	9.6	8.7
1	18.3	10.7
2	29.0	18.2
3	47.2	23.9
4	71.1	48.0
5	119.1	55.5
6	174.6	82.7
7	257.3	93.4
8	350.7	90.3
9	441.0	72.3
10	513.3	46.4
11	559.7	35.1
12	594.8	34.6
13	629.4	11.4
14	640.8	10.3
15	651.1	4.8
16	655.9	3.7
17	659.6	2.2
18	661.8	



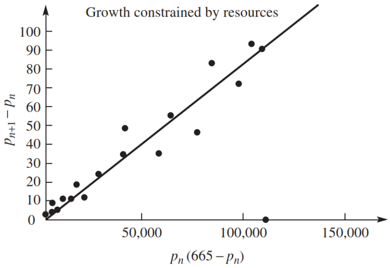
- The change increases with p_n for small p_n;
- The change decreases with p_n for large p_n;
- The yeast population approaches a limiting value, say 665.

Therefore, it is reasonable to propose the model

$$\Delta p_n = k(665 - p_n)p_n$$

That is, Δp_n is a linear function of $(665 - p_n)p_n$. Is it good?

$p_{n+1} - p_n$	$p_n(665 - p_n)$
8.7	6291.84
10.7	11,834.61
18.2	18,444.00
23.9	29,160.16
48.0	42,226.29
55.5	65,016.69
82.7	85,623.84
93.4	104,901.21
90.3	110,225.01
72.3	98,784.00
46.4	77,867.61
35.1	58,936.41
34.6	41,754.96
11.4	22,406.64
10.3	15,507.36
4.8	9050.29
3.7	5968.69
2.2	3561.84



They indeed have a linear relation (approximately).

How to find k? (see Chap. 3)

Suppose that $k = 0.00082$. Then our model is

$$\Delta p_n = 0.00082(665 - p_n)p_n$$

with $p_0 = 9.6$.

This dynamical system is called nonlinear because the right-hand side is a nonlinear function.

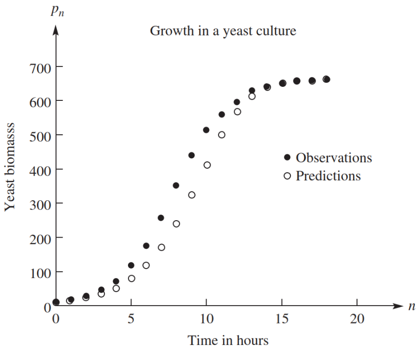
One can then use this model to predict the population.

$$\begin{aligned} p_1 &= p_0 + 0.00082(665 - p_0)p_0 \\ &= 9.6 + 0.00082(665 - 9.6)9.6 \\ &= 14.76. \end{aligned}$$

Other p_n can be found recursively.

Below is a comparison of observations and predictions.

Time in hours	Observation	Prediction
0	9.6	9.6
1	18.3	14.8
2	29.0	22.6
3	47.2	34.5
4	71.1	52.4
5	119.1	78.7
6	174.6	116.6
7	257.3	169.0
8	350.7	237.8
9	441.0	321.1
10	513.3	411.6
11	559.7	497.1
12	594.8	565.6
13	629.4	611.7
14	640.8	638.4
15	651.1	652.3
16	655.9	659.1
17	659.6	662.3
18	661.8	663.8



Hence, our model gives a satisfactory explanation of the yeast population.

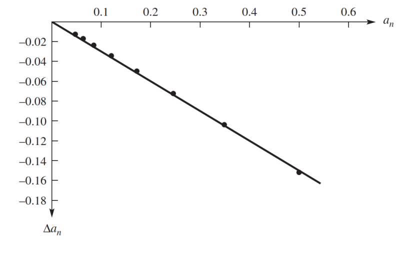
Example: decay of drug

Suppose your body has an initial dosage of 0.5 mg. The following is the amount of the drug measured at different time points.

a_n is the amount of drug at the end of n-th day.

n	0	1	2	3	4	5	6	7	8
a_n	0.500	0.345	0.238	0.164	0.113	0.078	0.054	0.037	0.026
Δa_n	−0.155	−0.107	−0.074	−0.051	−0.035	−0.024	−0.017	−0.011	

To find a model, we first look at the plot of Δa_n against a_n.



- It is reasonable to say that $\Delta a_n = ka_n$.
- We get $k = -0.31$ (see Chap. 3).

The model is

$$\Delta a_n = -0.31 a_n \quad \text{or} \quad a_{n+1} = 0.69 a_n$$

with $a_0 = 0.5$.

One can use the model to predict future drug amount.

Linear dynamical systems

We study the following

$$a_{n+1} = f(a_n) = ra_n + b$$

A number a is called an equilibrium value (EV) of a dynamical system if $a_0 = a$, then $a_n = a$ for all $n > 1$ (starts at a, solutions remain at a).

That is, $a_n = a$ for all n, is a constant solution.

Consequently, $a_{n+1} = f(a_n)$ implies that

$$a = f(a).$$

Thus, one can find a by solving the above equation.

We start with some examples.

Example: drug prescription

Assume that the drug is required to remain at a certain level. So, you need a certain daily dosage.

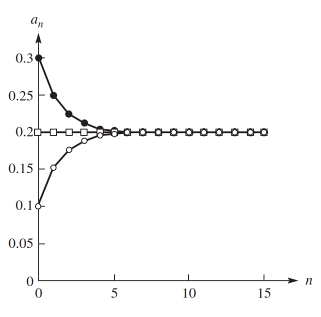
Assume that a daily dosage of 0.1 mg is used and it is known that half of the drug remains at the end of each day.

We obtain the model

$$a_{n+1} = 0.5a_n + 0.1.$$

We consider three starting values a_0

$$\begin{aligned} A: &\qquad a_0 = 0.1, \\ B: &\qquad a_0 = 0.2, \\ C: &\qquad a_0 = 0.3. \end{aligned}$$



- 0.2 is an equilibrium value (EV).
- If the initial dosage is above or below 0.2, the drug level will approach to 0.2.
- This is an evidence of stable equilibrium values (EVs).

The fact that 0.2 is a stable equilibrium value (EV) implies that the drug concentration will remain at 0.2 in the long run.

Example: annuity

A fixed amount is deposited initially and you are allowed to withdraw a fixed amount each month. An interesting issue is to determine how much you should deposit.

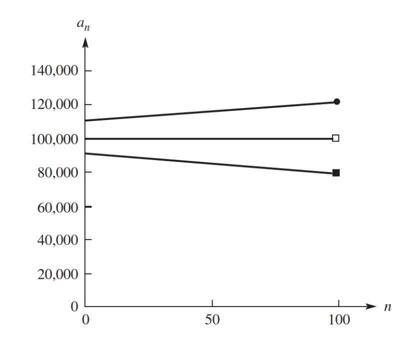
Assume that the interest rate is 1% per month and the monthly withdrawn is \$1000.

We obtain the model

$$a_{n+1} = 1.01a_n - 1000$$

We consider three initial deposits a_0

$$\begin{aligned} A: &\qquad a_0 = 90{,}000, \\ B: &\qquad a_0 = 100{,}000, \\ C: &\qquad a_0 = 110{,}000. \end{aligned}$$



- The value 100,000 is an equilibrium value (EV).
- If the initial deposit is above or below \$100,000, the amount will be moving away from 100,000.
- This is an evidence of unstable equilibrium values (EVs).

The fact that 100,000 is an unstable equilibrium value (EV) implies that your account will not be depleted if your initial deposit is more than \$100,000.

Classifying equilibrium values (EVs)

We have

$$a_{n+1} = ra_n + b.$$

If a is an equilibrium value (EV), then $a_{n+1} = a_n = a$. Then

$$a = ra + b.$$

Thus, if $r \neq 1$, then

$$a = \frac{b}{1 - r}.$$

Note:

- from the drug example, we have $r = 0.5$ and $b = 0.1$, then $a = 0.2$;
- from the annuity example, we have $r = 1.01$ and $b = -1000$, then $a = 100{,}000$.

Moreover, from the previous two examples, we conjecture that

- if $|r| < 1$, then a is a stable equilibrium value (EV);
- if $|r| > 1$, then a is an unstable equilibrium value (EV).

To see this, we note that the solution a_n can be expressed as

$$a_n = r^n c + \frac{b}{1-r},$$

where c is determined by a_0.

Hence, if $|r| < 1$, $a_n \to \frac{b}{1-r}$. If $|r| > 1$, a_n moves away from $\frac{b}{1-r}$.

To verify that

$$a_n = r^n c + \frac{b}{1-r}$$

is a solution, we first note that we should have

$$a_{n+1} = r^{n+1} c + \frac{b}{1-r}.$$

Then

$$\begin{aligned} ra_n + b &= r\left(r^n c + \frac{b}{1-r}\right) + b \\ &= r^{n+1} c + \frac{br}{1-r} + b \\ &= r^{n+1} c + \frac{br}{1-r} + \frac{b(1-r)}{1-r} \\ &= r^{n+1} c + \frac{b}{1-r} = a_{n+1}. \end{aligned}$$

This is mathematical induction.

Nonlinear dynamical systems

We study the following

$$a_{n+1} = f(a_n), \quad f \text{ is a nonlinear function.}$$

A number a is called an equilibrium value (EV) of a dynamical system

$$a = f(a).$$

To discuss stability, we use the Taylor expansion

$$f(y) = f(a) + (y-a)f'(a) + \mathcal{O}((y-a)^2).$$

Let $y = a_n$. We obtain

$$a_{n+1} - a = (a_n - a)f'(a) + \mathcal{O}((a_n - a)^2).$$

Erratum

Recall

$$a_{n+1} - a = (a_n - a)f'(a) + \mathcal{O}((a_n - a)^2)$$

Assume that a_n is close to a.

$$a_{n+1} - a = (a_n - a)(f'(a) + R),$$

where R is a small variable with $R = \mathcal{O}(|a_n - a|)$. Via the triangle inequality, we will have

$$|a_n - a|(|f'(a)| - |R|) \leq |a_{n+1} - a| \leq |a_n - a|(|f'(a)| + |R|).$$

Two cases:

- If $|f'(a)| < 1$, then $|f'(a)| + |R| < 1$. We have $a_n \to a$, it is a stable equilibrium value (EV).
- If $|f'(a)| > 1$, then $|f'(a)| - |R| > 1$. The sequence $\{a_n\}$ diverges, it is an unstable equilibrium value (EV).

System of difference equations

We consider modeling by a system of difference equations.

We are interested in the long-term behavior of the solutions.

If we start close to an equilibrium value, we want to know whether the solution will

- remain close,
- approach to the equilibrium value (EV),
- or not remain close.

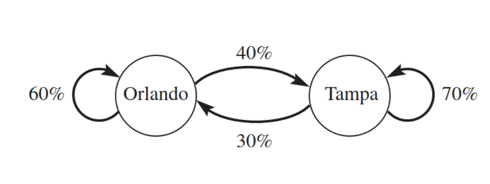
Moreover, we want to know whether this long-term behavior is sensitive to initial conditions.

Example: a rental car company

The company has two offices, Orlando and Tampa.

The rental cars can be returned in either city.

You want to know if there are sufficient cars in each city to meet the demand. If not, how many cars must be transferred from one city to another.

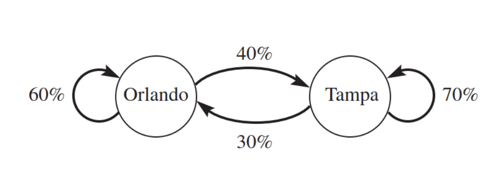


We will construct a model for this problem.

- O_n is the number of cars in Orlando at the end of day n,
- T_n is the number of cars in Tampa at the end of day n.

Then the historical record suggests the model

$$\begin{aligned} O_{n+1} &= 0.6\,O_n + 0.3\,T_n; \\ T_{n+1} &= 0.4\,O_n + 0.7\,T_n. \end{aligned}$$



We will find the equilibrium values (EVs) of

$$\begin{aligned} O_{n+1} &= 0.6\,O_n + 0.3\,T_n, \\ T_{n+1} &= 0.4\,O_n + 0.7\,T_n. \end{aligned}$$

Let (O, T) be an equilibrium value (EV). Then

$$O = O_n = O_{n+1}, \qquad T = T_n = T_{n+1}.$$

Hence, (O, T) satisfies

$$\begin{aligned} O &= 0.6\,O + 0.3\,T, \\ T &= 0.4\,O + 0.7\,T. \end{aligned}$$

The system has infinitely many solutions, namely, any (O, T) with $4O = 3T$ is a solution.

Now, assume that the company has 7000 cars. So, $O + T = 7000$.

Together with $4O = 3T$, we get

$$O = 3000, \qquad T = 4000,$$

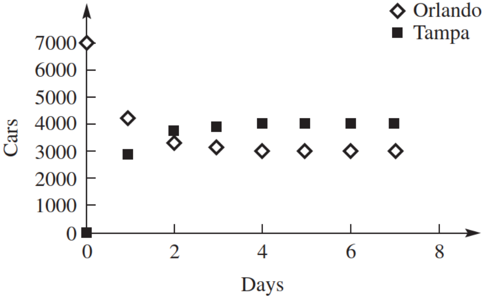
which is the equilibrium value.

Now, we try to see what happen for various initial conditions.

	Orlando	Tampa
Case 1	7000	0
Case 2	5000	2000
Case 3	2000	5000
Case 4	0	7000

$O_0 = 7000, \quad T_0 = 0.$

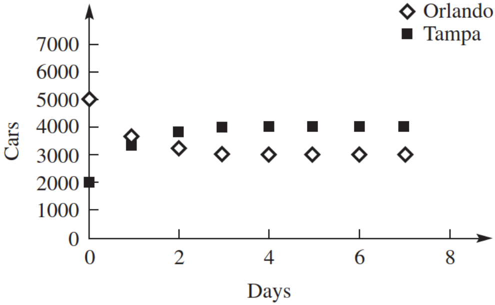
n	Orlando	Tampa
0	7000	0
1	4200	2800
2	3360	3640
3	3108	3892
4	3032.4	3967.6
5	3009.72	3990.28
6	3002.916	3997.084
7	3000.875	3999.125



a. Case 1

$O_0 = 5000, \quad T_0 = 2000.$

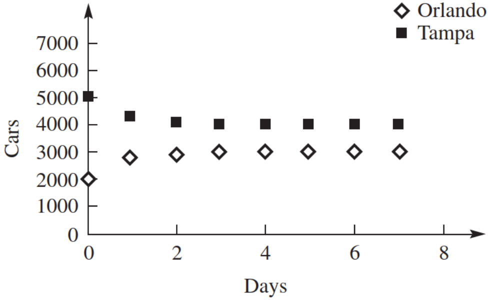
n	Orlando	Tampa
0	5000	2000
1	3600	3400
2	3180	3820
3	3054	3946
4	3016.2	3983.8
5	3004.86	3995.14
6	3001.458	3998.542
7	3000.437	3999.563



b. Case 2

$O_0 = 2000, \quad T_0 = 5000.$

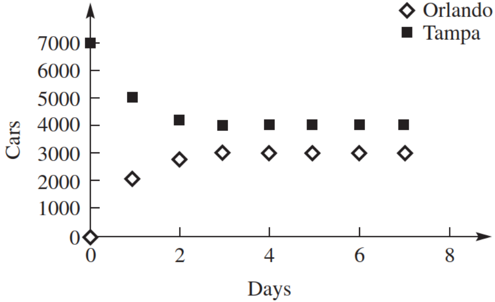
n	Orlando	Tampa
0	2000	5000
1	2700	4300
2	2910	4090
3	2973	4027
4	2991.9	4008.1
5	2997.57	4002.43
6	2999.271	4000.729
7	2999.781	4000.219



c. Case 3

$O_0 = 0, \quad T_0 = 7000.$

n	Orlando	Tampa
0	0	7000
1	2100	4900
2	2730	4270
3	2919	4081
4	2975.7	4024.3
5	2992.71	4007.29
6	2997.813	4002.187
7	2999.344	4000.656



d. Case 4

From the above calculations, we see that

- the number of cars approach to the equilibrium value (EV) (this is the long-term behavior);
- the equilibrium value (EV) is stable;
- the long-term behavior is insensitive to the starting values.

Knowing that 3000 cars will end up in Orlando and 4000 cars will end up in Tampa, the company can then decide its strategy.

Example: battle of Trafalgar

In 1805, French-Spanish naval force fought British naval force.

The French-Spanish had 33 ships and the British had 27 ships.

At each encounter, each side suffers from a loss equal to 10% of the number of ships of the opposing force.

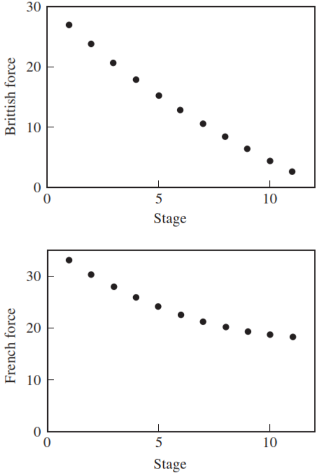
- B_n is the number of British ships at stage n,
- F_n is the number of French-Spanish ships at stage n.

Then we have the model

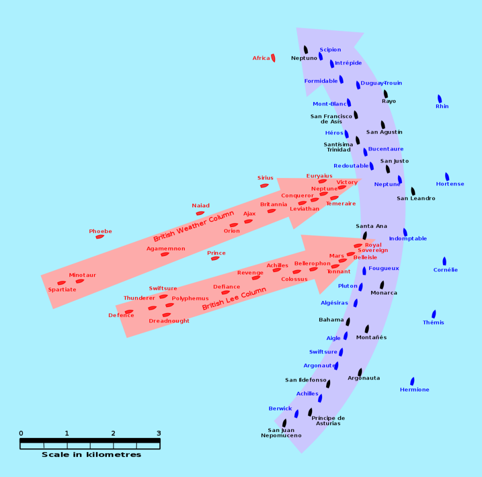
$$\begin{aligned} B_{n+1} &= B_n - 0.1\,F_n, \\ F_{n+1} &= F_n - 0.1\,B_n. \end{aligned}$$

Clearly, British force will lose.

Stage	Brittish force	French force
1	27.0000	33.0000
2	23.7000	30.3000
3	20.6700	27.9300
4	17.8770	25.8630
5	15.2907	24.0753
6	12.8832	22.5462
7	10.6285	21.2579
8	8.5028	20.1951
9	6.4832	19.3448
10	4.5488	18.6965
11	2.6791	18.2416



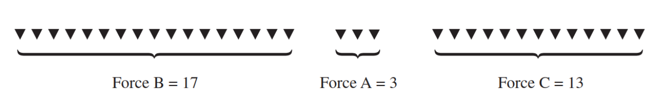
Nelson’s (British) Divide-and-Conquer strategy:



Divide British ships in 2 groups, and break the French-Spanish ships in 3 groups.

Then conquer one by one.

More precisely, French-Spanish ships are divided as follows.



- 13 ships from British force are engaged in Battle A;
- remaining ships (from Battle A) $+$ 14 ships ($14 = 27 - 13$) are engaged in Battle B;
- all remaining ships are engaged in Battle C.

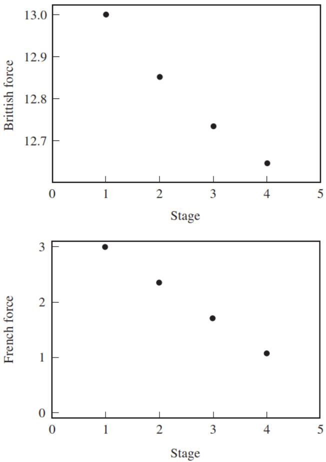
For display purpose, we assume each side loses 5% of the number of ships of the opposing force.

$$(B_{n+1} = B_n - 0.05F_n, \quad F_{n+1} = F_n - 0.05B_n)$$

British has 13 ships, French-Spanish has 3 ships.

Battle A

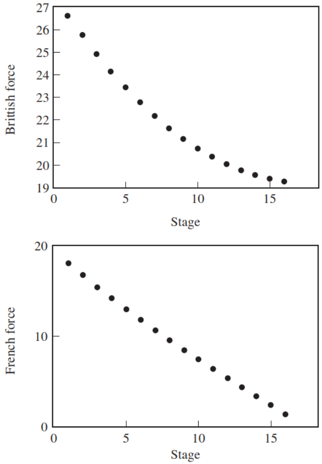
Stage	British force	French force
1	13.0000	3.00000
2	12.8500	2.35000
3	12.7325	1.70750
4	12.6471	1.07088



British has 12.6471 + 14 ships, French-Spanish has 1.07088 + 17 ships.

Battle B

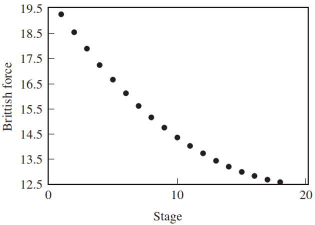
Stage	British force	French force
1	26.6471	18.0709
2	25.7436	16.7385
3	24.9066	15.4513
4	24.1341	14.2060
5	23.4238	12.9993
6	22.7738	11.8281
7	22.1824	10.6894
8	21.6479	9.5803
9	21.1689	8.4979
10	20.7440	7.4395
11	20.3720	6.4023
12	20.0519	5.3837
13	19.7827	4.3811
14	19.5637	3.3919
15	19.3941	2.4138
16	19.2734	1.4441

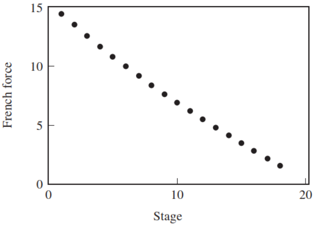


British has 19.2734 ships, French-Spanish has 1.4441 + 13 ships.

Battle C

Stage	British force	French force
1	19.2734	14.4441
2	18.5512	13.4804
3	17.8772	12.5529
4	17.2495	11.6590
5	16.6666	10.7965
6	16.1268	9.9632
7	15.6286	9.1569
8	15.1707	8.3754
9	14.7520	7.6169
10	14.3711	6.8793
11	14.0272	6.1607
12	13.7191	5.4594
13	13.4462	4.7734
14	13.2075	4.1011
15	13.0024	3.4407
16	12.8304	2.7906
17	12.6909	2.1491
18	12.5834	1.5146





British force won the battle (smartly divide your objective).

Example: competitive hunter model

Spotted owls and hawks compete for survival in a habitat.

Spotted owls

Hawks

Example: competitive hunter model

Spotted owls and hawks compete for survival in a habitat.

Assume that, in the absence of the other species, each individual species exhibits unconstrained growth in which the change is proportional to the population.

- O_n is the number of spotted owls at the end of day n,
- H_n is the number of hawks at the end of day n.

Then we have the model

$$\Delta O_n = k_1 O_n, \qquad \Delta H_n = k_2 H_n,$$

where k_1 and k_2 are positive constants.

The effect of the other species is to diminish the growth rate.

Note that there are many ways to model the mutual interaction of the two species. We assume that the decrease in the population is proportional to the product of the number of the two species.

Thus, we have the model

$$\begin{aligned}\Delta O_n &= k_1 O_n - k_3 O_n H_n,\\ \Delta H_n &= k_2 H_n - k_4 O_n H_n.\end{aligned}$$

That is

$$\begin{aligned}O_{n+1} &= (1+k_1)O_n - k_3 O_n H_n,\\ H_{n+1} &= (1+k_2)H_n - k_4 O_n H_n,\end{aligned}$$

where $k_1, \ldots, k_4$ are positive.

Now, we consider specific values of $k_1, \dots, k_4$. We have

$$\begin{aligned} O_{n+1} &= 1.2O_n - 0.001O_nH_n, \\ H_{n+1} &= 1.3H_n - 0.002O_nH_n. \end{aligned}$$

To find the equilibrium values (EVs) (O, H), we set $O = O_n = O_{n+1}$ and $H = H_n = H_{n+1}$,

$$\begin{aligned} O &= 1.2O - 0.001OH, \\ H &= 1.3H - 0.002OH. \end{aligned}$$

Then

$$\begin{aligned} 0 &= 0.2O - 0.001OH = O(0.2 - 0.001H), \\ 0 &= 0.3H - 0.002OH = H(0.3 - 0.002O). \end{aligned}$$

$$
\begin{aligned}
0 &= 0.2O - 0.001OH = O(0.2 - 0.001H). \\
0 &= 0.3H - 0.002OH = H(0.3 - 0.002O).
\end{aligned}
$$

From the first equation, we have

$$O = 0 \quad \text{or} \quad H = 200.$$

If $O = 0$, using the second equation, we have $H = 0$.

If $H = 200$, using the second equation, we have $O = 150$.

Hence, the two equilibrium values (EVs) are $(O, H) = (0, 0)$ and $(O, H) = (150, 200)$.

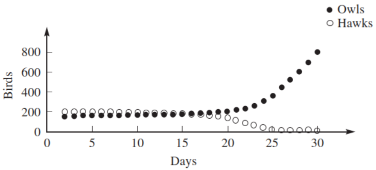
Now we consider the long-term behavior, consider the 3 cases.

	Owls	Hawks
Case 1	151	199
Case 2	149	201
Case 3	10	10

For Case 1 and Case 2, the initial values are close to the equilibrium value (EV) $(150, 200)$.

For Case 3, the initial value is close to the equilibrium value (EV) $(0, 0)$.

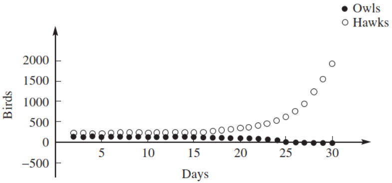
n	Owls	Hawks
1	151	199
2	151.151	198.602
3	151.3623	198.1448
4	151.6431	197.6049
5	152.0063	196.9556
6	152.4691	196.1653
7	153.0538	195.1966
8	153.7889	194.0044
9	154.711	192.5343
10	155.866	190.7202
11	157.3124	188.4827
12	159.1242	185.7261
13	161.3956	182.3369
14	164.2463	178.1812
15	167.83	173.1044
16	172.3438	166.9315
17	178.043	159.4717
18	185.2588	150.5276
19	194.424	139.9128
20	206.1064	127.4818
21	221.0528	113.1767
22	240.2454	97.09366
23	264.9681	79.56915
24	296.8785	61.27332
25	338.0634	43.27385
26	391.0468	26.99739
27	458.6989	13.98212
28	544.0252	5.349589
29	649.9199	1.133844
30	779.1669	0.000182



a. Case 1

Moving away from equilibrium value, hawks will die out.

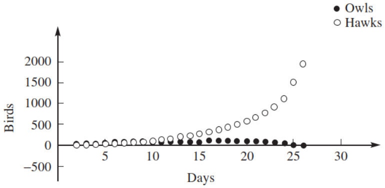
n	Owls	Hawks
1	149	201
2	148.851	201.402
3	148.6423	201.8648
4	148.3651	202.413
5	148.0071	203.0748
6	147.552	203.8842
7	146.9789	204.8824
8	146.2613	206.1204
9	145.3661	207.6616
10	144.2524	209.5862
11	142.8696	211.9954
12	141.1558	215.0186
13	139.0358	218.822
14	136.4189	223.6204
15	133.1966	229.6944
16	129.2414	237.4137
17	124.406	247.2705
18	118.5253	259.9277
19	111.4223	276.29
20	102.9219	297.6073
21	92.87598	325.6289
22	81.20808	362.8313
23	67.98486	412.751
24	53.52101	480.4547
25	38.51079	573.1623
26	24.14002	700.9651
27	12.04671	877.412
28	3.886124	1119.496
29	0.31285	1446.643
30	–0.07716	1879.731



b. Case 2

Moving away from equilibrium value, owls will die out.

n	Owls	Hawks
1	10	10
2	11.9	12.8
3	14.12768	16.33536
4	16.72244	20.77441
5	19.71952	26.31193
6	23.14457	33.16779
7	27.00583	41.58282
8	31.28402	51.81171
9	35.91994	64.11347
10	40.80098	78.7416
11	45.74844	95.93862
12	50.50908	115.9421
13	54.75477	139.0125
14	58.09413	165.493
15	60.09878	195.9126
16	60.34443	231.1382
17	58.46541	272.5838
18	54.22177	322.4855
19	47.58039	384.2597
20	38.81324	462.9712
21	28.60647	565.9237
22	18.13869	703.3227
23	9.009075	888.8048
24	2.803581	1139.432
25	0.169808	1474.872
26	−0.04668	1916.833



c. Case 3

Moving away from equilibrium value, owls will die out.

Conclusions

- If, initially, there are 151 owls and 199 hawks, then the hawks will die out.
- If, initially, there are 149 owls and 201 hawks, then the owls will die out.
- If, initially, there are 10 owls and 10 hawks, then the owls will die out.
- The equilibrium values (EVs) are unstable. If the starting values are close to an equilibrium value (EV), then the solutions will move away from the equilibrium value (EV).
- Long-term behavior is sensitive to initial conditions.
- This model predicts that coexistence is impossible.

Disclaimer

All figures, tables, and data appearing in the slides are only used for teaching under guidelines of Fair Use.