Math 3280 23-09-11

Review.

- Axiomatic approach to probability.

  A prob. P on the sample space S satisfies:

  Axiom I: $0 \le P(E) \le 1$, $\forall$ any Event E

  Axiom II: $P(S) = 1$.

  Axiom III: If $E_1, E_2, \cdots,$ is a sequence of events which are mutually exclusive,

  then $$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

  (Countable additivity of prob.).

- Properties derived from the above axioms:

  - $\mu(\phi) = 0$
  - (finite additivity) $\mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k)$ if $E_1, \cdots, E_n$ are disjoint
  - $\mu(E) = 1 - \mu(E^c)$
  - $\mu(E) \le \mu(F)$ if $E \subset F$

- $\mu(E\cup F) = \mu(E)+\mu(F) - \mu(E\cap F)$.

- (Countable sub-additivity of Prob)

$$P\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} P(E_k).$$

Prop.1 (Continuity of Probability)

① $P\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n)$ if $E_1 \subset E_2 \subset \cdots$

② $P\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n)$ if $E_1 \supset E_2 \supset \cdots$

Pf. We first prove ①.

Write $F_1 = E_1$

$F_2 = E_2 \setminus E_1$

$\cdots$

$F_n = E_n \setminus \bigcup_{i=1}^{n-1} E_i$

$\cdots$

Then $F_1, \cdots, F_n, \cdots$ are mutually exclusive.

and $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i = E_n$

$\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$

Hence

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = P\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} P(F_n)$$ (since (F_n) are mutually disjoint)

$$= \lim_{n\to\infty} \sum_{k=1}^{n} P(F_k)$$

$$= \lim_{n\to\infty} P(F_1 \cup \cdots \cup F_n)$$ (since $F_1, \cdots, F_n$ are disjoint)

$$= \lim_{n\to\infty} P(E_n).$$

Next we prove ②.

Notice that

$$E_1^c \subset E_2^c \subset \cdots$$

By ①,

$$P\left(\bigcup_{n=1}^{\infty} E_n^c\right) = \lim_{n\to\infty} P(E_n^c).$$

But

$$\text{LHS} = 1 - P\left(\bigcap_{n=1}^{\infty} E_n\right),$$

$$\text{RHS} = \lim_{n\to\infty} 1 - P(E_n).$$

This implies that $P\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n).$

Example 2. If $P(E) = 0.8$, $P(F) = 0.9$

Show that $P(E \cap F) \geq 0.7$.

Pf. Recall

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Hence

$$\begin{aligned} P(E \cap F) &= P(E) + P(F) - P(E \cup F) \\ &= 0.8 + 0.9 - P(E \cup F) \\ &\geq 0.8 + 0.9 - 1 = 0.7. \quad \blacksquare \end{aligned}$$

Example 3.

If $P(E) = 0.8$, $P(F) = 0.9$, $P(E \cap F) = 0.75$

find the probability that exactly one of E and F occurs.

Solution: Let H denote the event that exactly one of E and F occurs.

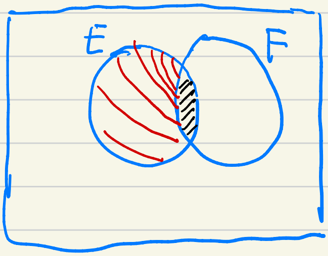
Then

$$H = (E \setminus F) \cup (F \setminus E)$$

(disjoint union)

Hence $P(H) = P(E \setminus F) + P(F \setminus E)$.

Notice that $E = (E \setminus F) \cap (E \cap F)$



Hence $P(E) = P(E \setminus F) + P(E \cap F)$

It follows that

$$P(E \setminus F) = P(E) - P(E \cap F)$$

$$= 0.8 - 0.75$$

$$= 0.05.$$

Similarly,

$$P(F \setminus E) = P(F) - P(E \cap F)$$

$$= 0.9 - 0.75$$

$$= 0.15$$

Hence

$$P(H) = P(E \setminus F) + P(F \setminus E)$$

$$= 0.05 + 0.15$$

$$= 0.20.$$

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