Math 3280 A 2023/11/2

Review:

- Joint CDF of X and Y:

$$F(a,b) = P\{X \le a, Y \le b\}, \quad a, b \in \mathbb{R}.$$

- All joint probability statements about X and Y are determined by the joint CDF of X and Y.

Next we further study the joint distributions in the discrete case and continuous case, separately.

(1) Discrete case.

- Now we consider the case that both X and Y are discrete. In such case, we can define the joint prob. mass function of X and Y by
  (joint pmf)
  $$p(x,y) = P\{X=x, Y=y\}.$$

Then

$$\begin{aligned} p_X(x) &= P\{X=x\} \\ &= \sum_y P\{X=x, Y=y\} \\ &= \sum_y p(x,y). \end{aligned}$$

similarly

$$p_Y(y) = \sum_x p(x,y).$$

In particular

$$F(a,b) = \sum_{\substack{(x,y) \\ x\le a,\, y\le b.}} p(x,y) \,.$$

(2) Continuous case.

- Def: We say two r.v's X and Y are jointly continuous if there exists $f: \mathbb{R}^2 \to [0, \infty)$ such that

$$P\{(X,Y) \in C\} = \iint_C f(x,y)\,dx\,dy,$$

for any "measurable" set $C \subset \mathbb{R}^2$.

("measurable" sets include, for instance, the countable union/intersections of rectangles $[a,b] \times [c,d]$)

- In particular,

$$P\{X \leq a, Y \leq b\} = P\{(X,Y) \in (-\infty, a) \times (-\infty, b)\}$$
$$= \int_{-\infty}^{b} \int_{-\infty}^{a} f(x,y)\,dx\,dy.$$

- The function f in the def is called the joint prob. density function of X and Y.

Prop 1. Suppose X and Y have a joint density f.
Let F be the joint CDF of X and Y, and let f_X and f_Y be the marginal densities of X and Y.

Then

(1) $$\frac{\partial^2 F(a,b)}{\partial a \partial b} = f(a,b) \quad \text{for} \quad a, b \in \mathbb{R}.$$

(2) $$f_X(a) = \int_{-\infty}^{\infty} f(a,y)\,dy, \qquad a \in \mathbb{R}$$

$$f_Y(b) = \int_{-\infty}^{\infty} f(x,b)\,dx, \qquad b \in \mathbb{R}.$$

Pf. (1) Recall that

$$F(a,b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{a} g(x)\,dx \quad \text{where } g(x) = \int_{-\infty}^{b} f(x,y)\,dy$$

So $$\frac{\partial F(a,b)}{\partial a} = g(a) = \int_{-\infty}^{b} f(a,y)\,dy$$

and $$\frac{\partial^2 F(a,b)}{\partial a \partial b} = \frac{\partial}{\partial b} \int_{-\infty}^{b} f(a,y)\,dy = f(a,b).$$

(2)

$$F_X(a) = P\{X \leq a\}$$
$$= \int_{-\infty}^{a} \left(\int_{-\infty}^{\infty} f(x,y)\,dy \right) dx$$

Let $h(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

Then

$$F_X(a) = \int_{-\infty}^{a} h(x)\,dx$$

Taking derivatives gives

$$f_X(a) = \frac{d\,F_X(a)}{da} = h(a) = \int_{-\infty}^{\infty} f(a,y)\,dy.$$

Similarly

$$f_Y(b) = \int_{-\infty}^{\infty} f(x,b)\,dx.$$

$\blacksquare$

Example 2. Suppose X and Y have a joint density function

$$f(x,y) = \begin{cases} 12\, xy\,(1-x) & \text{if } 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$$

Find f_X and $E[X]$.

Solution: By Prop 1,

$$f_X(a) = \int_{-\infty}^{\infty} f(a,y)\, dy.$$

For $a \in (0,1)$,

$$f_X(a) = \int_0^1 12a(1-a)\,y\, dy$$

$$= 6a(1-a).$$

For $a \notin (0,1)$, $f_X(a) = 0$.

Hence $f_{X(a)} = \begin{cases} 6a(1-a) & \text{if } a \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$

Now

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_0^1 6x^2(1-x)\, dx$$

$$= \int_0^1 6x^2 - 6x^3\, dx$$

$$= 2x^3 - \frac{3}{2}x^4 \Big|_0^1$$

$$= \frac{1}{2}.$$

Example 3.

Suppose X and Y have a joint denstity function

$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{if } 0<x<\infty,\ 0<y<\infty \\ 0 & \text{otherwise.} \end{cases}$$

Find the prob. density function of $\frac{X}{Y}$.

Solution:

Since $f(x,y)=0$ if $(x,y)\notin(0,\infty)\times(0,\infty)$, we may assume X, Y always take positive values. So is X/Y.

For $a>0$,

$$P\left\{\frac{X}{Y}\le a\right\} = P\{X\le aY\}$$

$$= \iint_{\{(x,y):\ x\le ay\}} f(x,y)\,dx\,dy$$

$$= \iint_{\{(x,y)\in(0,\infty)\times(0,\infty)\,:\, x\le ay\}} e^{-(x+y)} \, dx\, dy$$

$$= \int_0^\infty \int_0^{ay} e^{-(x+y)} \, dx\, dy$$

$$= \int_0^\infty e^{-y} \cdot \left(-e^{-x}\right)\Big|_0^{ay} \, dy$$

$$= \int_0^\infty e^{-y} \cdot \left(1 - e^{-ay}\right) dy$$

$$= \int_0^\infty e^{-y} - e^{-(a+1)y} \, dy$$

$$= -e^{-y} + \frac{1}{1+a} e^{-(a+1)y} \Big|_0^\infty$$

$$= 1 - \frac{1}{1+a}$$

Taking derivative gives

$$f_{\frac{X}{Y}}(a) = \begin{cases} \frac{1}{(1+a)^2} & a > 0 \\ 0 & \text{otherwise}. \end{cases}$$

$\boxtimes$

§6.2 Independent random variables

Recall that two events E and F are said to be independent if $P(E \cap F) = P(E)P(F)$.

Def: Let X and Y be two r.v.'s.

We say that X and Y are independent if

$$P\{X \in A, Y \in B\} = P\{X \in A\} P\{Y \in B\},$$

for all $A, B \subset \mathbb{R}$. That is, the events $\{X \in A\}$ and $\{Y \in B\}$ are independent for all $A, B \subset \mathbb{R}$.

Remark: X and Y are independent

$$\Longleftrightarrow$$

$$F(a,b) = F_X(a) F_Y(b), \quad \forall a, b \in \mathbb{R}.$$

The direction "$\Rightarrow$" is clear. The other direction can be proved by using the three axioms of probability.

- Equivalent def of independence for r.v.'s.

Prop 5. Suppose X and Y are discrete. Then

X and Y are independent

$$\Longleftrightarrow \quad p(x,y) = p_X(x)\, p_Y(y) \qquad (*)$$

Pf. Clearly X and Y are independent

$$\Longleftrightarrow \quad P\{X\in A, Y\in B\} = P\{X\in A\}\cdot P\{Y\in B\}.$$

Letting $A=\{x\}$, $B=\{y\}$ gives

$$p(x,y) = p_X(x)\, p_Y(y).$$

Now suppose $(*)$ holds for all x, y,

Then for given $A, B \subset \mathbb{R}$,

$$\begin{aligned} P\{X\in A, Y\in B\} &= \sum_{x\in A}\sum_{y\in B} p(x,y) \\ &= \sum_{x\in A}\sum_{y\in B} p_X(x)\, p_Y(y) \\ &= \Big(\sum_{x\in A} p_X(x)\Big)\Big(\sum_{y\in B} p_Y(y)\Big) \\ &= P\{X\in A\}\, P\{Y\in B\}. \quad \square \end{aligned}$$

Prop 6. If X and Y are jointly continuous, then X and Y are independent

$$\Leftrightarrow \quad f(x,y) = f_X(x) f_Y(y).$$

Pf. X and Y are independent

$$\Leftrightarrow \quad F(a,b) = F_X(a)\, F_Y(b), \quad \forall\, a,b \in \mathbb{R}$$

$$\Rightarrow \quad \frac{\partial^2 F(a,b)}{\partial a\, \partial b} = \frac{d\, F_X(a)}{d\, a} \cdot \frac{d\, F_Y(b)}{d\, b}$$

i.e. $\quad f(a,b) = f_X(a)\, f_Y(b). \quad (**)$.

Now if $(**)$ holds, then

$$F(a,b) = \int_{-\infty}^{b} \int_{-\infty}^{a} f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{b} \int_{-\infty}^{a} f_X(x) f_Y(y)\, dx\, dy$$

$$= \left(\int_{-\infty}^{b} f_Y(y)\, dy\right) \left(\int_{-\infty}^{a} f_X(x)\, dx\right)$$

$$= F_Y(b) \cdot F_X(a).$$

Hence X, Y are independent. $\square$

Example 7: Suppose X and Y have a joint density

$$f(x,y) = 24xy, \quad \text{if } 0<x<1,\ 0<y<1,\ 0<x+y<1.$$

Determine whether X and Y are independent.

<u>Solution</u>: We first calculate the marginal densities $f_X(x)$, $f_Y(y)$.

Notice that for $0<a<1$,

$$f(a,y) = \begin{cases} 24ay, & \text{if } 0<y<1-a, \\ 0 & \text{otherwise.} \end{cases}$$

So
$$f_X(a) = \int_{-\infty}^{-\infty} f(a,y)\,dy$$

$$= \int_0^{1-a} 24ay\,dy$$

$$= 24a \left.\frac{y^2}{2}\right|_0^{1-a} = 12a\cdot(1-a)^2$$

Similarly,

$$f_Y(b) = \int_{-\infty}^{\infty} f(x,b)\,dx$$

$$= \int_0^{1-b} 24\,x\,b\,dx$$

$$= 12\,b\,(1-b)^2 \qquad \text{if } 0<b<1 .$$

Clearly $f(a,b) \neq f_X(a)\,f_Y(b)$. Hence X, Y are not independent. $\blacksquare$