Math 3280A 2023/10/30

Review.

- Normal r.v X with parameters μ and σ^2:

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}, \qquad x \in (-\infty, \infty)$$

$$Z := \frac{X-\mu}{\sigma} \text{ is a standard normal r.v.}$$

(with mean 0, Variance 1)

- Exponential r.v with parameter λ $(\lambda > 0)$,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

§ 5.7 The distribution of a function of a cts r.v

Q: Let X be a cts r.v. with density f_X

Let $g: \mathbb{R} \to \mathbb{R}$ be a function.

Let $Y = g(X)$.

How to find the distribution of Y?

Example 1. Let X be a cts r.v. with density f_X.

Let $Y = X^2$.

Find the pdf of Y.

Solution: We first calculate the CDF of Y:

$$
\begin{aligned}
F_Y(a) &= P\{Y \leq a\} \\
&= P\{X^2 \leq a\} \\
&= \begin{cases} 0 & \text{if } a < 0 \\ P\{-\sqrt{a} \leq X \leq \sqrt{a}\} & \text{if } a \geq 0 \end{cases} \\
&\qquad = F_X(\sqrt{a}) - F_X(-\sqrt{a})
\end{aligned}
$$

Taking derivative of F_Y with respect to a gives

$$
f_Y(a) = \begin{cases} 0 & \text{if } \quad a \leq 0 \\ \dfrac{f_X(\sqrt{a}) + f_X(-\sqrt{a})}{2\sqrt{a}} & \text{if } \quad a > 0 \end{cases}
$$

$\square$

Prop. 2 : Let X be a cts r.v. with density f_X. Let $g: \mathbb{R} \to \mathbb{R}$ be a differentiable, strictly monotone function. Let $Y = g(X)$, then the pdf of Y is given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \left| \dfrac{d\, g^{-1}(y)}{dy} \right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{otherwise.} \end{cases}$$

pf. WLOG assume that g is strictly monotone increasing.

First suppose that $y = g(x)$ for some x. Then

$$F_Y(y) = P\{ Y \leq y \}$$
$$= P\{ g(X) \leq y \}$$
$$= P\{ X \leq g^{-1}(y) \}$$
$$= F_X(g^{-1}(y)).$$

Taking derivative with respect to y gives

$$f_Y(y) = F_Y'(y) = f_X(g^{-1}(y)) \cdot \frac{d\, g^{-1}(y)}{dy}.$$

If $y \notin \text{range}(g)$, then either

$$F_Y(y) = 0 \text{ or } 1.$$

Taking derivative gives

$$f_Y(y) = 0.$$

$\square$

Example 3. Let Z be a standard normal r.v. Find the density of Z^3.

Solution: Let $g(x) = x^3$. Then g is monotone increasing on $\mathbb{R}$ with $\text{range}(g) = \mathbb{R}$.

Hence

$$f_{Z^3}(z) = f_Z(g^{-1}(z)) \cdot \frac{d\, g^{-1}(z)}{dz}$$

$$= f_Z(z^{1/3}) \cdot \frac{1}{3} \cdot z^{-2/3}$$

$$= \frac{1}{3\sqrt{2\pi}} \cdot z^{-2/3} \cdot e^{-\frac{z^{2/3}}{2}}.$$

$\blacksquare$

Chap 6. Joint distributed r.v.'s.

§ 6.1 Joint distribution.

Def. Let X, Y be two r.v.'s.

The joint cumulative distribution function of X, Y (joint CDF) is defined by

$$F(a,b) = P\{X \le a, Y \le b\}, \quad a, b \in \mathbb{R}.$$

From the above def, we see that F_X and F_Y can be obtained from $F(\cdot,\cdot)$.

$$\begin{aligned}
F_X(a) &= P\{X \le a\} \\
&= P\{X \le a, Y < \infty\} \\
&= P\left(\lim_{b\to+\infty} \{X \le a, Y \le b\}\right) \\
&\overset{\text{Using the continuity of } P}{=} \lim_{b\to+\infty} P\{X \le a, Y \le b\} \\
&= \lim_{b\to+\infty} F(a,b) \\
&=: F(a,+\infty)
\end{aligned}$$

Similary, $F_Y(b) = \lim_{a\to+\infty} F(a,b) =: F(+\infty, b)$.

- Usually, F_X and F_Y are called the marginal distributions of X and Y.

Theoretically, all the joint statements about X and Y can be determined by the function $F(a,b)$.

Example 4. Suppose $F = F(a,b)$ is the joint CDF of X and Y. Find $P\{X > a, Y > b\}$.

Solution:

$$P\{X>a, Y>b\} = 1 - P(\{X>a, Y>b\}^c)$$

$$= 1 - P(\{X \leq a\} \cup \{Y \leq b\})$$

(using $P(E \cup F) = P(E) + P(F) - P(EF)$)

$$= 1 - P\{X \leq a\} - P\{Y \leq b\} + P\{X \leq a, Y \leq b\}$$

$$= 1 - F(a, \infty) - F(\infty, b) + F(a,b).$$