Math 3280A 23-09-24

Review.

- Independence of 2 or more events
- Independence of sub-experiments.
- Random variables, discrete random variables.
- Prob. mass function and expected value of a discrete r.v.

§ 4.4. Expected value of a function of a r.v.

Let $X: S \to \mathbb{R}$ be a discrete r.v.

Let $g: \mathbb{R} \to \mathbb{R}$ be a function.

Then $g(X)$ is a function from S to $\mathbb{R}$,

so it is a new r.v.

Q: How can we compute $E[g(X)]$?

Remark: By def, if $y_1, y_2, \cdots,$ are the possible values of $g(X)$, then

$$E[g(X)] = \sum_j y_j P\{g(X) = y_j\}.$$

Below we give a short-cut formula for $E[g(X)]$.

Prop. $E[g(X)] = \sum_i g(x_i) \cdot P(x_i)$,

where $x_1, x_2, \cdots$, are all the possible values of X.

Pf. Grouping all $g(x_i)$ which take the same value, gives

$$\sum_i g(x_i)\, p(x_i)$$

$$= \sum_j \left(\sum_{i:\, g(x_i)=y_j} g(x_i)\, p(x_i) \right)$$

$$= \sum_j \left(\sum_{i:\, g(x_i)=y_j} y_j\, P(x_i) \right)$$

$$= \sum_j y_j \left(\sum_{i:\, g(x_i)=y_j} P(x_i) \right)$$

$$= \sum_j y_j\, P(g(X) = y_j) \qquad (*)$$

To see (*), Notice that

$$\{ g(X) = y_j \} = \bigcup_{i : g(x_i) = y_j} \{ X = x_i \}$$

(the Union being disjoint)

Hence
$$P\{ g(X) = y_j \} = \sum_{i : g(x_i) = y_j} P\{ X = x_i \}$$
$$= \sum_{i : g(x_i) = y_j} p(x_i).$$

$\square$

<u>Corollary</u>: $E[aX+b] = aE[X] + b$

$\forall\ a, b \in \mathbb{R}$

and X is a discrete r.v.

Pf. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = ax+b$.

$$\text{LHS} = E[g(X)] = \sum_{x : p(x) > 0} (ax+b)\, p(x)$$

$$= \sum_{x : p(x) > 0} ax\,p(x) + b\,p(x)$$

$$= a \sum_{x : p(x) > 0} x\,p(x) + b \sum_{x : p(x) > 0} p(x)$$

$$= a \cdot E[X] + b = \text{RHS}.$$ $\square$

Def. Let X be a discrete r.v.

For each non-negative integer n,

we call $E[X^n]$ the n-th moment of X.

§ 4.5 Variance.

Def. Let X be a discrete r.v.

Set

$$Var(X) = E[(X-\mu)^2], \text{ where } \mu = E[X].$$

we call $Var(X)$ the variance of X.

It describes how far X is spread out from its mean.

Prop. $\mathrm{Var}(X) = E[X^2] - \mu^2$.

Pf. By definition

$$\mathrm{Var}(X) = E[(X-\mu)^2]$$

$$= \sum_{x: p(x)>0} (x-\mu)^2 p(x)$$

$$= \sum_{x: p(x)>0} x^2 p(x) - 2\mu \sum_{x: p(x)>0} x\, p(x)$$

$$+ \mu^2 \sum_{x: p(x)>0} p(x)$$

$$= E[X^2] - 2\mu^2 + \mu^2$$

$$= E[X^2] - \mu^2.$$

$\blacksquare$