Math 3280 23-11-30

Review

Thm (The weak law of large numbers)

Let $X_1, X_2, \cdots, X_n, \cdots$ be an i.i.d sequence of r.v's, having a finite mean $\mu$. Then for any $\varepsilon > 0$,

$$P\left\{ \left| \frac{X_1 + \cdots + X_n}{n} - \mu \right| \geq \varepsilon \right\} \to 0 \text{ as } n \to \infty.$$

Thm (The central limit Thm).

Let $X_1, \cdots, X_n, \cdots$, be an i.i.d. sequence of r.v.'s, each having finite mean $\mu$ and variance $\sigma^2$.

Then $\forall\, a \in \mathbb{R}$,

$$P\left\{ \frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\,\sigma} \leq a \right\} \longrightarrow \Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{\frac{-x^2}{2}}\, dx,$$

as $n \to \infty$.

To prove the CLT, we state a result without proof.

<u>Lem 1</u>. Let $Z_1, \cdots, Z_n, \cdots$ be a sequence of r.v's with distribution functions $F_{Z_n}$. Let $Z$ be a r.v. with distribution function $F_Z$.

Suppose $M_{Z_n}(t) \to M_Z(t)$ for all $t \in \mathbb{R}$ as $n \to \infty$. (Recall $M_Z(t) := E[e^{tZ}]$)

Then
$$F_{Z_n}(t) \to F_Z(t) \quad \text{for each } t \text{ at}$$
which $F_Z$ is cts, as $n \to \infty$.

<u>Pf of the CLT.</u>

First assume $\mu = 0$, $\sigma^2 = 1$.

Let
$$Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}, \quad n = 1, 2, \cdots.$$

Let $Z$ be the standard normal r.v.

Recall $M_Z(t) = e^{t^2/2}$, $t \in \mathbb{R}$.

Hence we only need to prove for $t \in \mathbb{R}$,

① $M_{Z_n}(t) \longrightarrow e^{t^2/2}$ as $n \to \infty$.

Notice that

$$M_{Z_n}(t) = E\left[ e^{t \cdot \frac{X_1 + \cdots + X_n}{\sqrt{n}}} \right]$$

$$= \prod_{j=1}^{n} E\left[ e^{t X_i / \sqrt{n}} \right]$$

$$= \left( M_X\left( \frac{t}{\sqrt{n}} \right) \right)^n, \quad \text{where } X = X_1$$

To show ①, it is equivalent to show

② $n \log M_X\left(\frac{t}{\sqrt{n}}\right) \to t^2/2$ as $n \to \infty$.

For convenience, we write

$$L(t) = \log M_X(t).$$

Clearly, $L(0) = 0$.

Notice that

$$L'(t) = \frac{M_X'(t)}{M_X(t)}, \quad L''(t) = \frac{M_X''(t) M_X(t) - \left(M_X'(t)\right)^2}{M_X(t)^2}$$

In particular

$$L'(0) = \frac{M_X'(0)}{M_X(0)} = \frac{E[X]}{1} = \mu = 0$$

$$L''(0) = \frac{M_X''(0) \cdot M_X(0) - M_X'(0)^2}{M_X(0)^2} = \frac{E[X^2]}{1}$$

$$= Var(X) + E[X]^2$$

$$= 1$$

Hence

$$\lim_{n \to \infty} n L\left(\frac{t}{\sqrt{n}}\right) = \lim_{n \to \infty} \frac{L\left(\frac{t}{\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}}\right)^2}$$

$$\overset{\text{Letting } x = \frac{1}{\sqrt{n}}}{=} \lim_{x \to 0} \frac{L(tx)}{x^2}$$

$$\underline{\underline{\text{L'Hopital's rule}}} \quad \lim_{x \to 0} \frac{L'(tx) \cdot t}{2x}$$

$$= \lim_{x \to 0} \frac{L''(tx)\, t^2}{2}$$

$$= \frac{t^2}{2} L''(0) = t^2/2 .$$

In the general case,

$$\frac{X_1+\cdots+X_n-n\mu}{\sqrt{n}\cdot\sigma}=\frac{\frac{X_1-\mu}{\sigma}+\cdots+\frac{X_n-\mu}{\sigma}}{\sqrt{n}}$$

Notice that $\widetilde{X}_i=\frac{X_i-\mu}{\sigma}$ has mean 0 and variance 1

Since $\widetilde{X}_1,\cdots,\widetilde{X}_n,\cdots,$ are i.i.d with mean 0 and variance 1, the distribution of

$$\frac{\widetilde{X}_1+\cdots+\widetilde{X}_n}{\sqrt{n}}$$ converges to the standard normal distribution. $\square$

**Thm 3** (The strong law of large numbers).

Let $X_1, \cdots, X_n, \cdots,$ be an i.i.d. sequence of r.v.'s with a finite mean $\mu$. Then with prob. 1,

$$\frac{X_1 + \cdots + X_n}{n} \longrightarrow \mu \qquad \text{as } n \to \infty.$$

In other word,

$$P\left\{ \lim_{n\to\infty} \frac{X_1+\cdots+X_n}{n} = \mu \right\} = 1.$$

**Lem 4.** Assume $X$ is a non-negative r.v. which may take the value $+\infty$. Suppose that $E[X] < \infty$. Then $P\{X < \infty\} = 1$.

Pf. $P\{X = \infty\} \le P\{X \ge n\} \le \frac{E[X]}{n} \to 0.$ ▨

**Pf of Thm 3 (SLLN):**

We will the thm under an additional assumption

$$E[X_i^4] = K < \infty.$$

WLOG, assume $\mu = 0$.

Write $S_n = X_1 + \cdots + X_n$.

We will estimate

$$E[S_n^4] = E[(X_1 + \cdots + X_n)^4].$$

Expand $(X_1 + \cdots + X_n)^4$ in terms of

$$X_i^4, \quad X_i^3 X_j, \quad X_i^2 X_j^2, \quad X_i^2 X_j X_k, \quad X_i X_j X_k X_\ell$$

with distinct $i, j, k, \ell$.

Notice that $E[X_i^3 X_j] = E[X_i^3] E[X_j] = 0$

$$E[X_i^2 X_j X_k] = E[X_i^2] E[X_j] E[X_k] = 0$$

$$E[X_i X_j X_k X_\ell] = 0$$

Hence

$$E[S_n^4] = E[(X_1 + \cdots + X_n)^4]$$

$$= n E[X_i^4]$$

$$+ \binom{n}{2}\binom{4}{2} E[X_i^2 X_j^2]$$

$$= n E[X_i^4] + 6\binom{n}{2} E[X_i^2] E[X_i^2]$$

(Using an inequality

$$E[X^2]^2 \leq E[X^4]$$

( reason: $\mathrm{Var}(X^2) = E[X^4] - E[X^2]^2 \geq 0$ )

$$E[S_n^4] \leq n E[X_i^4] + 6\binom{n}{2} E[X_i^4]$$

$$= \left(n + 6 \cdot \frac{n(n-1)}{2}\right) K$$

$$= (3n^2 - 2n) K.$$

$$\leq 3n^2 K.$$

$$E\left[\frac{S_n^4}{n^4}\right] \leq \frac{3K}{n^2}.$$

Hence

$$\sum_{n=1}^{\infty} E\left[\left(\frac{S_n}{n}\right)^4\right] \leq \sum_{n=1}^{\infty} \frac{3K}{n^2} < \infty.$$

Thus

$$E\left[\sum_{n=1}^{\infty}\left(\frac{S_n}{n}\right)^4\right] = \sum_{n=1}^{\infty} E\left[\left(\frac{S_n}{n}\right)^4\right] < \infty.$$

Let $X = \sum_{n=1}^{\infty}\left(\frac{S_n}{n}\right)^4$. Then $X$ is a r.v, non-negative (may take the value $\infty$)

However $E[X] < \infty$

By Lem 4, $P\{X < \infty\} = 1$.

Hence $P\left\{\sum_{n=1}^{\infty}\left(\frac{S_n}{n}\right)^4 < \infty\right\} = 1$

However $\sum_{n=1}^{\infty}\left(\frac{S_n}{n}\right)^4 < \infty \Rightarrow \frac{S_n}{n} \to 0$ as $n \to \infty$.

Hence $P\left\{\lim_{n\to\infty} \frac{S_n}{n} = 0\right\} = 1$.

Hence with Prob. 1,

$$\frac{S_n}{n} = \frac{X_1+\cdots+X_n}{n} \to 0 .$$

If $\mu \neq 0$, then letting $\widetilde{X}_n = X_n - \mu$

applying the SLLN to $(\widetilde{X}_n)$ gives

$$\frac{\widetilde{X}_1+\cdots+\widetilde{X}_n}{n} \to 0 \qquad \text{almost sure.}$$

$$\Longleftrightarrow \quad \frac{X_1+\cdots+X_n}{n} \to \mu \qquad \text{almost sure.}$$

$\boxed{\text{A}}$