Math 3280 23-11-23

Review

- Conditional expectation $E[X|Y=y]$
- Calculate expectation by conditioning

$$E[X] = E[E[X|Y]].$$

§ 7.7. Moment generating functions.

Def. Let X be a r.v. and $t \in \mathbb{R}$. Define

$$M(t) = E[e^{tX}]$$

and we call $M(t)$ the moment generating function of X

Remark: • Since

$$e^{tX} = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n,$$

we have

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E[X^n] \quad (1)$$

- $M(0) = 1$.

- For some random variables X, $M(t)$ may not exist for some $t \in \mathbb{R}$.

For instance

$$P\{X = n\} = \frac{1}{n(n+1)}, \quad n = 1, 2, \cdots$$

Then $E[X^k] = \infty \quad \forall\ k = 1, 2, \cdots,$ so

$$E(e^{tX}) = \infty \quad \forall\ t > 0.$$

Prop 1. Suppose $M(t)$ exists and is finite on a neighborhood $(-t_0, t_0)$ of 0, then

$$M^{(n)}(0) = E[X^n], \quad n = 1, 2, \cdots$$

Example 2. Let X be a binomial r.v. with parameters (n, p). Calculate $M(t)$.

$$M(t) = E[e^{tX}] = \sum_{k=0}^{n} e^{tk} \cdot \binom{n}{k} p^k \cdot (1-p)^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} (e^t p)^k (1-p)^{n-k}$$
$$= (e^t p + (1-p))^k$$

Example 3. Let X be the Poisson r.v. with parameter λ. Calculate $M(t)$.

Solution.
$$M(t) = E[e^{tX}]$$
$$= \sum_{n=0}^{\infty} e^{tn} \cdot \frac{\lambda^n}{n!} e^{-\lambda}$$
$$= \sum_{n=0}^{\infty} \frac{(e^t\lambda)^n}{n!} e^{-\lambda}$$
$$= e^{e^t\lambda} \cdot e^{-\lambda}$$
$$= e^{\lambda(e^t-1)}$$

Example 4. Let X be a standard normal r.v. Calculate $M(t)$.

Solution:
$$M(t) = E[e^{tX}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \cdot e^{\frac{-x^2}{2}} dx$$
$$= \frac{1}{\sqrt{2\pi}} e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}} dx$$

$$= e^{t^2/2}.$$

Example 5. Let X be a normal r.v. with mean μ and Variance.

Calculate $M(t)$ for X.

Solution: Let $Z = \frac{X-\mu}{\sigma}$. Then Z is a standard normal r.v.

Hence

$$M(t) = E[e^{tX}] = E[e^{t(\mu+\sigma Z)}]$$

$$= e^{t\mu} \cdot E[e^{t\sigma Z}]$$

$$= e^{t\mu} \cdot e^{t^2\sigma^2/2}$$

$$= e^{\frac{t^2\sigma^2}{2}+t\mu}.$$

Prop 6. If X, Y are independent, then

$$M_{X+Y}(t) = M_X(t)\, M_Y(t).$$

Pf. $M_{X+Y}(t) = E[e^{tX+tY}]$

$$= E[e^{tX} \cdot e^{tY}]$$

$$= E[e^{tX}] \cdot E[e^{tY}].$$

$$= M_X(t)\, M_Y(t)$$

Thm 7. If $M_X(t) = M_Y(t)$ on a neighborhood $(-t_0, t_0)$ of zero, then X, Y have the same (cumulative) distribution

In this sense, we say that the moment generating function uniquely determines the distribution.

Chap 8. Limiting thms.

§8.1 Introduction

Q: Let $X_1, X_2, \cdots,$ be a sequence of independent, identically distributed r.v.'s. What can we say about the limiting behavior of

$$\frac{X_1 + \cdots + X_n}{n} \quad \text{as } n \to \infty \text{ ?}$$

§8.2. Two basic inequalities

Prop 8. (Markov inequality)

Let X be a non-negative r.v. Then for any $a > 0$,

$$P\{X \geq a\} \leq E[X]/a.$$

Pf. Let $I = \begin{cases} 1 & \text{if } X \geq a \\ 0 & \text{otherwise.} \end{cases}$

Then I is a r.v. so that $I \leq \frac{X}{a}$

(here we use the fact $X \geq 0$)

So $E[I] \leq E[\frac{X}{a}]$

But $E[I] = 1 \cdot P\{I=1\} + 0 \cdot P\{I=0\}$

$= P\{X \geq a\}$.

Hence $P\{X \geq a\} \leq E[X]/a$.

Prop 9. (Chebyshev's inequality)

Let X be a r.v. with finite mean μ and variance σ^2.
Then for any $\varepsilon > 0$,

$$P\{\, |X-\mu| \geq \varepsilon \,\} \leq \sigma^2/\varepsilon^2$$

Pf. Let $Y = |X-\mu|^2$. Applying Markov inequality to Y gives

$$P\{\, |X-\mu| \geq \varepsilon \} = P\{\, Y \geq \varepsilon^2 \} \leq \frac{E[Y]}{\varepsilon^2}$$

$$\leq \quad E[(X-\mu)^2] / \varepsilon^2$$

$$= \quad Var(X)/\varepsilon^2 \quad = \quad \sigma^2/\varepsilon^2.$$

Example.

Suppose that it is known that the number of items produced in a factory during a week is a random variable with mean 50.

(a) What can be said about the probability that this week's production will exceed 75?

(b) If the variance of a week's production is known to equal 25, then what can be said about the probability that this week's production will be between 40 and 60?

Solution: Let X be the number of items produced in a week. $E[X] = 50$

Then

(a) By Markov, $P\{X > 75\} \leq \dfrac{E[X]}{75} = \dfrac{2}{3}$.

(b) Since $\sigma^2 = 25$,

$$P\{40 < X \leqslant 60\} = P\{|X-50| \leqslant 10\}$$

$$\geqslant 1 - \frac{\sigma^2}{10^2}$$

$$\geqslant 1 - \frac{25}{100} = \frac{3}{4} = 0.75.$$

$\square$.