Math 3280 23-11-20

Review.

- $Cov(X,Y) = E[(X-E[X])(Y-E[Y])]$
$$= E[XY] - E[X]E[Y].$$

- $Cov(X,Y) = 0$ if X, Y are independent.

Prop 1. (1) $Cov(X,Y) = Cov(Y,X)$.

(2) $Cov(X,X) = Var(X)$.

(3) $Cov(aX, Y) = a\,Cov(X,Y)$, $a \in \mathbb{R}$.

(4) $$Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov(X_i, Y_j).$$

(1), (3), (4) imply that $Cov(\cdot,\cdot)$ is bi-linear.

Pf. Let us prove (4) only. Write $\mu_i = E[X_i]$,
$\nu_i = E[Y_j]$.

Then by definition,

$$\mathrm{Cov}\left(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j\right) = E\left[\left(\sum_{i=1}^n X_i - \sum_{i=1}^n \mu_i\right)\cdot\left(\sum_{j=1}^m Y_j - \sum_{j=1}^m \nu_j\right)\right]$$

$$= E\left[\sum_{i=1}^n (X_i - \mu_i) \sum_{j=1}^m (Y_j - \nu_j)\right]$$

$$= E\left[\sum_{i=1}^n \sum_{j=1}^m (X_i - \mu_i)(Y_j - \nu_j)\right]$$

By the linearity of E

$$= \sum_{i=1}^n \sum_{j=1}^m E\left[(X_i - \mu_i)(Y_j - \nu_j)\right]$$

$$= \sum_{i=1}^n \sum_{j=1}^m \mathrm{Cov}(X_i, Y_j).$$

$\square$

Corollary 2. $$\mathrm{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathrm{Var}(X_i) + 2\cdot\sum_{1\le i<j\le n} \mathrm{Cov}(X_i, X_j).$$

Moreover if $X_1, \ldots, X_n$ are pairwise independent, then

$$\mathrm{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathrm{Var}(X_i).$$

§ 7.5 Conditional expectations.

Def. If X and Y are discrete, then

the conditional expectation of X given $Y=y$, is

$$E[X \mid Y=y] := \sum_x x \; P\{X=x \mid Y=y\}$$

provided that $P\{Y=y\} > 0$.

Def. In the case when X and Y are jointly cts with a density $f(x,y)$, the conditional expectation of X given $Y=y$, is defined by

$$E[X \mid Y=y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\, dx,$$

provided that $f_Y(y) > 0$, where

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}.$$

Example 1. Let X, Y be jointly cts with a density

$$f(x,y) = \begin{cases} e^{-x/y} \cdot e^{-y}/y & \text{if } x, y > 0, \\ 0 & \text{otherwise}. \end{cases}$$

Calculate $E[X \mid Y = y]$, $y > 0$.

Solution:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^{\infty} e^{-x/y} e^{-y}/y \; dx$$

$$= -e^{-x/y} e^{-y} \Big|_{x=0}^{\infty}$$

$$= e^{-y}, \quad \text{if } y > 0$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$= e^{-x/y}/y \quad \text{if } x, y > 0.$$

$$E[X|Y=y] = \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y) \, dx$$

$$= \int_{0}^{\infty} x \cdot e^{-x/y} / y \, dx$$

$$\overset{\text{Int by Part}}{=} x \cdot \left(-e^{-x/y}\right) \Big|_{x=0}^{+\infty} + \int_{0}^{\infty} e^{-x/y} \cdot dx$$

$$= 0 + \left(-y e^{-x/y}\right) \Big|_{x=0}^{+\infty}$$

$$= y \quad \text{if} \quad y > 0.$$

$\blacksquare$

Now write

$E[X|Y]$ as a function of Y by

$$y \longmapsto E[X|Y=y]$$

$E[X|Y]$ is a r.v., the value of which depends on the value of Y.

Prop 2. $E[X] = E[\,E[X|Y]\,]$

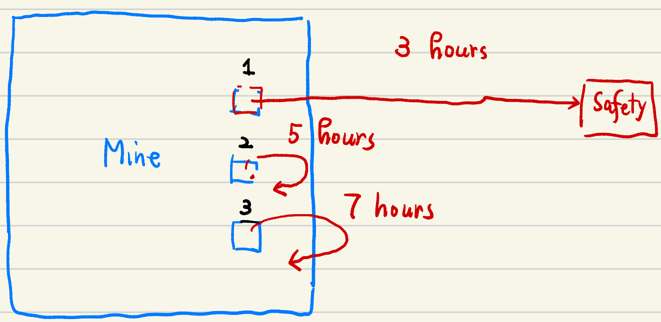
Pf. We only prove it in the discrete case.

$$\begin{aligned}
E[\,E[X|Y]\,] &= \sum_y E[X|Y=y] \cdot P_Y(y) \\
&= \sum_y \cdot \sum_x x \cdot P\{X=x|Y=y\} \cdot P_Y(y) \\
&= \sum_y \sum_x x\, P\{X=x, Y=y\} \\
&= \sum_x \sum_y x\, P\{X=x, Y=y\} \\
&= \sum_x x\, P\{X=x\} = E[X]
\end{aligned}$$

$\square$

Example 3.

A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours. If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he reaches safety?



Solution: Let X denote the length of time (in hours) until the miner reaches safety.

Let Y denote the door that he choose in the first time.

By Prop 2,

$$E[X] = E[\,E[X|Y]\,]$$

$$= E[X|Y=1]\cdot P\{Y=1\}$$

$$+ E[X|Y=2]\cdot P\{Y=2\}$$

$$+ E[X \mid Y=3] \cdot P\{Y=3\}$$

$$= \frac{1}{3}\Big(E[X \mid Y=1] + E[X \mid Y=2] + E[X \mid Y=3]\Big)$$

$$= \frac{1}{3}\Big(3 + (5 + E[X]) + (7 + E[X])\Big)$$

Solving this equation, we obtain

$$E[X] = 3 + 5 + 7 = 15 \quad \text{(hours)}.$$

$\blacksquare$