

## Review.

- Conditional distribution.

$X, Y$  are jointly cts.

$$\textcircled{1} f_{X|Y}(x|y) := \frac{f_{(X,Y)}(x,y)}{f_Y(y)} \quad \text{if } f_Y(y) > 0$$

$$\textcircled{2} P\{X \in A | Y=y\} := \int_A f_{X|Y}(x|y) dx$$

- Distribution of functions of r.v.'s.

Setup:  $X_1, X_2$  are joint cts with density

$f_{X_1, X_2}(x_1, x_2)$ . Let  $g_1, g_2: \mathbb{R}^2 \rightarrow \mathbb{R}$ .

Let  $Y_1 = g_1(X_1, X_2)$ ,  $Y_2 = g_2(X_1, X_2)$ .

Find out the joint distribution of  $Y_1, Y_2$

Thm. Assumptions: ①  $x_1, x_2$  can be solved  
in terms of  $y_1, y_2$ .

②  $g_1, g_2$  have cts partial derivatives

and

$$J(x_1, x_2) := \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$$
$$= \frac{\partial g_1}{\partial x_1} \frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2} \frac{\partial g_2}{\partial x_1} \neq 0$$

Then  $Y_1, Y_2$  have a joint density

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(x_1, x_2) \cdot |J(x_1, x_2)|^{-1}$$

Exer 1.  $X, Y$  have joint density

$$f(x, y) = \begin{cases} \frac{1}{x^2 y^2}, & \text{if } x > 1, y > 1 \\ 0, & \text{otherwise.} \end{cases}$$

Let  $U = XY, V = X/Y$ . Find out the joint density  
of  $U$  and  $V$ .

Solution: Let  $g_1(x, y) = xy$  and  $g_2(x, y) = x/y$ .

Then

$$J(x, y) = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} \\ = -\frac{2x}{y}$$

By the thm,

$$f_{U, V}(u, v) = f(x, y) \cdot |J(x, y)|^{-1} \\ = \begin{cases} \frac{1}{x^2 y^2} \cdot \frac{y}{2x} = \frac{1}{2x^3 y} & \text{if } x > 1, y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that

$$x = \sqrt{uv}$$

$$y = \sqrt{u/v}$$

$$u, v > 0.$$

$$\begin{cases} \sqrt{uv} > 1 \Rightarrow v > \frac{1}{u} \\ \sqrt{u/v} > 1 \Rightarrow v < u \end{cases}$$

Also notice that  $x, y > 1 \Leftrightarrow u > 1, \frac{1}{u} < v < u$ .

$$\text{Hence } f_{U, V}(u, v) = \begin{cases} \frac{1}{2u^2 v}, & \text{if } u > 1, \frac{1}{u} < v < u \\ 0, & \text{otherwise.} \end{cases} \quad \square$$

## Chap. 7. Properties of expectations.

### § 7.1 Introduction.

Recall that in the discrete case

$$E[X] = \sum_x x p(x).$$

In the cts case,

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

The expectation of  $X$  is a weighted average of the possible values of  $X$ .

## § 7.2 Expectation of functions of r.v.'s and sums of r.v.'s

Prop 2. Let  $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ .

(1) If both  $X$  and  $Y$  are discrete with a joint prob. mass function  $p(x, y)$ , then

$$E[g(X, Y)] = \sum_x \sum_y g(x, y) p(x, y)$$

(2) If  $X, Y$  are jointly cts with a density  $f(x, y)$ , then

$$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dx dy$$

pf. It is similar to the one r.v. case. □

Corollary 3.  $E[X+Y] = E[X] + E[Y]$ .

pf. We only prove the result in the case when

$X, Y$  are jointly cts.

$$\begin{aligned}\text{Then by Prop. 2, } E[X+Y] &= \iint_{-\infty}^{\infty} (x+y) f_{(X,Y)} dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{(X,Y)} dx dy \\ &\quad + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{(X,Y)} dx dy \\ &= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} x f_{(X,Y)} dy \right) dx \\ &\quad + \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} y f_{(X,Y)} dx \right) dy \\ &= \int_{-\infty}^{\infty} x f_X(x) dx + \int_{-\infty}^{\infty} y f_Y(y) dy \\ &= E[X] + E[Y]. \quad \square\end{aligned}$$

By induction, we have the following

Corollary 4.

$$E[X_1 + \dots + X_n] = \sum_{i=1}^n E[X_i].$$

## § 7.4 Covariance.

Recall the variance of a r.v.  $X$  is given by

$$\text{Var}(X) = E[(X-\mu)^2], \text{ where } \mu = E[X].$$

It describes how far is  $X$  from its mean.

Def. (Covariance)

Let  $X, Y$  be two r.v.'s. The covariance of  $X$  and  $Y$ , denoted by  $\text{Cov}(X, Y)$ , is defined by

$$\text{Cov}(X, Y) = E[(X - E[X]) \cdot (Y - E[Y])].$$

In particular,  $\text{Cov}(X, X) = \text{Var}(X)$ .

Lem 4. Let  $X, Y$  be independent, and  $g, h: \mathbb{R} \rightarrow \mathbb{R}$ .

Then

$$E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)].$$

Pf. We only prove it in the cts case.

$$E[g(X)h(Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y) f(x, y) dx dy$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) h(y) f_X(x) f_Y(y) dx dy \\
&= \left( \int_{-\infty}^{\infty} g(x) f_X(x) dx \right) \left( \int_{-\infty}^{\infty} h(y) f_Y(y) dy \right) \\
&= E[g(X)] \cdot E[h(Y)].
\end{aligned}$$

Corollary 5. If  $X, Y$  are independent,  
then  $\text{Cov}(X, Y) = 0$ .

pf. When  $X, Y$  are independent, by Lem 4,

$$\begin{aligned}
\text{Cov}(X, Y) &= E[(X - E[X])(Y - E[Y])] \\
&= E[(X - E[X])] \cdot E[(Y - E[Y])] \\
&= 0
\end{aligned}$$

□

Remark:  $\text{Cov}(X, Y) = 0$  does not imply that  $X, Y$  are independent.

Example 6. Let  $X, Y$  be two r.v.'s such that

$$\textcircled{1} P\{X=0\} = P\{X=-1\} = P\{X=1\} = \frac{1}{3}$$

$$\textcircled{2} Y = \begin{cases} 0 & \text{if } X \neq 0. \\ 1 & \text{if } X = 0. \end{cases}$$

• A short cut formula

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

$$\bullet E[X] = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot (1) = 0$$

$$P\{XY=0\} = 1 \Rightarrow E[XY] = 0 \cdot 1 = 0$$

Hence  $\text{Cov}(X, Y) = 0$ .

But  $X, Y$  are not independent.

$$P\{X=0, Y=0\} = 0$$

$$\text{But } P\{X=0\} = \frac{1}{3}$$

$$P\{Y=0\} = P\{X \neq 0\}$$

$$= P\{X=1\} + P\{X=-1\}$$

$$= \frac{2}{3}$$

$$\text{Hence } P\{X=0, Y=0\} \neq P\{X=0\} \cdot P\{Y=0\}$$

Therefore,  $X$  and  $Y$  are not independent.