Math 3280 A 23-11-06

Review.

- Joint distribution of two r.v.'s
- Independence of two r.v.'s.

Remark: The above concepts can be generalized to finitely many r.v.'s.

Let $X_1, X_2, \cdots, X_n$ be r.v.'s.

- The joint CDF of $X_1, \cdots, X_n$ is given by

$$F(a_1, a_2, \cdots, a_n) = P\{X_1 \leq a_1, X_2 \leq a_2, \cdots, X_n \leq a_n\}.$$

$$a_1, \cdots, a_n \in \mathbb{R}.$$

- We say that $X_1, \cdots, X_n$ are jointly continuous if, $\exists\ f: \mathbb{R}^n \to [0, \infty)$ such that for any "measurable" set $C \subset \mathbb{R}^n$,

$$P\{(X_1, \cdots, X_n) \in C\} = \int \cdots \int_C f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n.$$

- We say that $X_1, \cdots, X_n$ are independent if

$$P\{X_1 \in A_1, X_2 \in A_2, \cdots, X_n \in A_n\}$$

(*)

$$= P\{X_1 \in A_1\} \cdots P\{X_n \in A_n\},$$

for all $A_1, \cdots, A_n \subset \mathbb{R}$.

Theoretically, one can prove that (*) is equivalent to

(**) $F(a_1, a_2, \cdots, a_n) = F_{X_1}(a_1) \cdots F_{X_n}(a_n), \ \forall\, a_1, \cdots, a_n \in \mathbb{R}$.

§ 6.3 Sums of independent r.v.'s.

Question: Let X, Y be independent r.v.'s.

How to calculate the distribution of $X+Y$?

1. The case that both X and Y are continuous.

Let X, Y have densities $f_X(x)$, $f_Y(y)$ resp.

Since X and Y are assumed to be independent,

X, Y have a joint density

$$f(x,y) = f_X(x)\, f_Y(y).$$

Now let $a \in \mathbb{R}$, then

$$F_{X+Y}(a) = P\{X+Y \le a\}$$

$$= \iint_{\substack{(x,y)\in\mathbb{R}^2: \\ x+y \le a}} f(x,y)\, dx\, dy$$

$$= \iint_{(x,y)\in\mathbb{R}^2:\ x+y\le a} f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{a-y} f_X(x)\, f_Y(y)\, dx \right) dy$$

$$= \int_{-\infty}^{\infty} f_Y(y) \cdot \left(\int_{-\infty}^{a-y} f_X(x)\, dx \right) dy$$

$$= \int_{-\infty}^{\infty} f_Y(y)\, F_X(a-y)\, dy$$

$$=: \quad F_X * f_Y(a)$$

Convolution

(For $g, h : \mathbb{R} \to \mathbb{R}$, we let

$$g * h(a) = \int_{-\infty}^{\infty} g(a-y)\, h(y)\, dy$$)

$$f_{X+Y}(a) = \frac{d}{da} F_{X+Y}(a)$$

$$= \frac{d}{da} \int_{-\infty}^{\infty} F_X(a-y)\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \left(\frac{d}{da} F_X(a-y) \right) f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy$$

$$= f_X * f_Y(a).$$

Thm A. If X, Y are independent, and have densities $f_X(x)$, $f_Y(y)$, then

$$\begin{cases} F_{X+Y}(a) = F_X * f_Y(a). \\ f_{X+Y}(a) = f_X * f_Y(a). \end{cases}$$

Example 1. Let X, Y be independent, both unif. dist. on $[0,1]$. Calculate the density of $X+Y$.

Solution: Let $a \in \mathbb{R}$. Then

$$\begin{aligned} f_{X+Y}(a) &= f_X * f_Y(a) \\ &= \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy \\ &= \int_0^1 f_X(a-y)\, dy \quad \left(\text{since } f_Y(y) = \begin{cases} 1 & \text{if } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}\right) \\ &\overset{\text{letting } z=a-y}{=\!=} \int_{a-1}^{a} f_X(z)\, dz \end{aligned}$$

If $0 < a \leq 1$,

$$\int_{a-1}^{a} f_X(z)\,dz = \int_{0}^{a} 1\,dz = a.$$

If $1 < a \leq 2$,

$$\int_{a-1}^{a} f_X(z)\,dz = \int_{a-1}^{1} 1\,dz = 2-a.$$

If $a > 2$ or $a < 0$,

$$\int_{a-1}^{a} f_X(z)\,dz = 0.$$

Hence

$$f_{X+Y}(a) = \begin{cases} a & \text{if } 0 < a \leq 1 \\ 2-a & \text{if } 1 < a \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

$\blacksquare$

Example 2. Let X, Y be independent normal r.v's with parameters $(0,1)$ and $(0,\sigma^2)$. Find out the distribution of $X+Y$.

Solution: Recall that

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{y^2}{2\sigma^2}}, \quad y \in \mathbb{R}.$$

Hence for $a \in \mathbb{R}$,

$$f_X * f_Y(a) = \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(a-y)^2}{2}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{y^2}{2\sigma^2}}\, dy$$

$$= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} e^{-\frac{(a-y)^2}{2} - \frac{y^2}{2\sigma^2}}\, dy.$$

Notice that

$$\frac{(a-y)^2}{2} + \frac{y^2}{2\sigma^2} = \frac{\left(\sqrt{\sigma^2+1}\, y - \frac{a\sigma^2}{\sqrt{\sigma^2+1}}\right)^2}{2\sigma^2} + \frac{a^2}{2(\sigma^2+1)}$$

(verify it)

Hence

$$f_{X+Y}(a) = \frac{1}{2\pi\sigma} e^{-\frac{a^2}{2(\sigma^2+1)}} \int_{-\infty}^{\infty} e^{-\frac{\left(\sqrt{\sigma^2+1}\, y - \frac{a\sigma^2}{\sqrt{\sigma^2+1}}\right)^2}{2\sigma^2}} dy$$

(letting $z = \dfrac{\sqrt{\sigma^2+1}\, y - \frac{a\sigma^2}{\sqrt{\sigma^2+1}}}{\sigma}$)

$$= \frac{1}{2\pi\sigma} \cdot \frac{\sigma}{\sqrt{\sigma^2+1}} e^{-\frac{a^2}{2(\sigma^2+1)}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz$$

(Using the fact $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1$)

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2+1}} e^{-\frac{a^2}{2(\sigma^2+1)}}.$$

Hence $X+Y$ is a normal r.v. with parameters $(0, \sigma^2+1)$.

Remark: In general, if X, Y are independent, normal r.v.'s with parameters (μ_1, σ_1^2), and (μ_2, σ_2^2), then $X+Y$ has a normal distribution with parameters $(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.