Math 3280A 23-10-05

Review

- Bernoulli r.v. with parameter p
- Binomial r.v with parameters (n, p).
- Poisson r.v. with parameter λ.

§ 4.9 Expectation of sums of discrete r.v.'s

Let $X_1, X_2, \cdots, X_n$ be discrete r.v.'s on the same sample space S.

Prop 1. $E[X_1 + \cdots + X_n] = \sum_{k=1}^{n} E[X_k]$.

We will prove the above result under an additional assumption that S is finite or countably infinite. (The general case is referred to Theoretical Exer 4.36 in the text book).

Lem 2. Assume that S is finite or countably infinite.

Set $p(s) = P(\{s\})$ for $s \in S$.

Then for any r.v. X on S, we have

$$E[X] = \sum_{s \in S} X(s)\, p(s).$$

Pf. Suppose the distinct values of X are x_i, $i \geq 1$.

Let $S_i = \{ s \in S : X(s) = x_i \}$.

Then $S_1, S_2, \cdots,$ are a partition of S.

By definition,

$$E[X] = \sum_i x_i P\{X = x_i\}$$

$$= \sum_i x_i P(S_i)$$

$$= \sum_i x_i \sum_{s \in S_i} p(s)$$

$$= \sum_i \sum_{s \in S_i} x_i p(s)$$

$$= \sum_i \sum_{s \in S_i} X(s) p(s)$$

$$= \sum_{s \in S} X(s)\, p(s)$$

(because $S = \bigcup_i S_i$ with the Union being disjoint) $\blacksquare$

<u>Pf of Prop 1.</u>

By Lem 2,

$$E[X_1 + \cdots + X_n] = \sum_{s \in S} \left(X_1(s) + \cdots + X_n(s) \right) p(s)$$

$$= \left(\sum_{s \in S} X_1(s)\, p(s) \right) + \cdots + \left(\sum_{s \in S} X_n(s)\, p(s) \right)$$

$$= E[X_1] + \cdots + E[X_n].$$

$\blacksquare$.

§ 4.9. Cumulative distribution function.

Def. Let X be a discrete r.v. Define

$$F_X(b) = P\{X \leq b\}, \quad b \in \mathbb{R}.$$

We call F_X the cumulative distribution function (CDF) of X. We also write $F(b) = F_X(b)$.

Prop 3. (1) F is non-decreasing, that is

$$F(a) \leq F(b) \quad \text{if} \quad a < b.$$

(2) $\lim\limits_{b \to +\infty} F(b) = 1$.

(3) $\lim\limits_{b \to -\infty} F(b) = 0$.

(4) F is right continuous, i.e.

$$\lim_{b_n \downarrow b} F(b_n) = F(b).$$

(i.e. b_n tends to b from the RHS of b)

The prop. is based on the continuity property of probability

Recall that if $E_n \nearrow E$, then $\lim P(E_n) = P(E)$

(i.e $E_{n+1} \supset E_n$, $E = \bigcup_{n=1}^{\infty} E_n$)

if $E_n \searrow E$ ($E_{n+1} \subset E_n$, $E = \bigcap_{n=1}^{\infty} E_n$)

then $P(E_n) \to P(E)$ as $n \to \infty$.

pf of Prop 3.

(1) Since if $a < b$, it follows that

$$\{X \leq a\} \subset \{X \leq b\}.$$

So $F(a) \leq F(b)$.

(2) If $b_n \nearrow \infty$,

then $\{X \leq b_n\} \nearrow \{X < \infty\} = S$

So $F(b_n) \to 1$ (by the continuity property of probability)

(3) If $b_n \searrow -\infty$, then

$$\{X \leq b_n\} \searrow \{X = -\infty\} = \emptyset$$

So $F(b_n) \to 0$.

(4) If $b_n \searrow b$, then

$$\{X \leq b_n\} \searrow \{X \leq b\}.$$

So $\lim_{n \to \infty} F(b_n) = F(b)$.

Thus F is right continuous.

<u>Remark</u>: • In general, F is not left continuous.

Prop 4. $P\{X=b\} = F(b) - F(b-)$

Pf. Let $b_n \nearrow b$ with $b_n < b$.

Then $\{X \le b_n\} \nearrow \{X<b\}$.

So $P\{X \le b_n\} \to P\{X<b\}$.

Thus $P\{X<b\} = \lim_{n\to\infty} F(b_n) = F(b-)$.

It follows that

$$P\{X=b\} = P\{X\le b\} - P\{X<b\}$$
$$= F(b) - F(b-). \quad \blacksquare$$

Chap 5. Continuous r.v.'s.

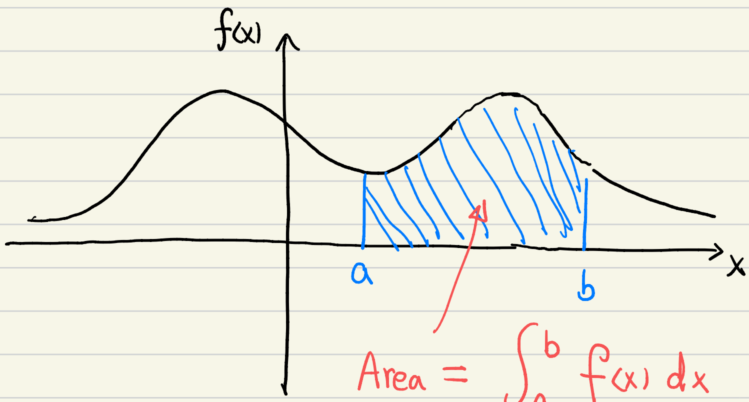
§ 5.1 Introduction.

Def. X is called a cont. r.v. (or absolute cont. r.v.) if there is a non-negative function defined on $(-\infty, \infty)$ such that

$$P\{X \in B\} = \int_B f(x)\,dx$$

for all "measurable" sets $B \subset (-\infty, \infty)$.

Remark: The measurable sets include all intervals, and the countable unions / intersections of intervals.



$$\text{Area} = \int_a^b f(x)\,dx$$

$$P\{a \le X \le b\} = \int_a^b f(x)\,dx$$

$$= \text{Area of the shaded region.}$$

- We call f the prob. density function (pdf) of X.

Example 2. Let X be a cts r.v. with pdf

$$f(x) = \begin{cases} C(4x - 2x^2) & \text{if } x \in (0,2) \\ 0 & \text{otherwise.} \end{cases}$$

(1) Find the value of C

(2) Find $P\{X \le 1\}$.

- $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1.$

Since $P\{X \in (-\infty, \infty)\} = P\{S\} = 1$

Solution:

$$1 = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_0^2 C\left(4x - 2x^2\right) dx$$

$$= C \cdot \left(2x^2 - \frac{2}{3}x^3\right)\Big|_0^2$$

$$= C \cdot \left(8 - \frac{2}{3} \times 8\right) = \frac{8}{3} C .$$

Hence $C = \frac{3}{8}$.

$$P\{X \le 1\} = \int_{-\infty}^{1} f(x)\,dx$$

$$= \int_0^1 \frac{3}{8}\left(4x - 2x^2\right) dx$$

$$= \frac{3}{8}\left(2x^2 - \frac{2}{3}x^3\right)\Big|_0^1 .$$

$$= \frac{1}{2} .$$