Recall

Common discrete distributions

Bernoulli r.v.. with parameter $p \in [0,1]$. Denote $X \sim Bern(p)$.

(1) X is the outcome of a trial that succeeds with probability p and fails with probability $1-p$.

(2) The PMF of X is $p(1) = P\{X=1\} = p$ and $p(0) = P\{X=0\} = 1-p$.

Note $E[X] = p$ and $\text{Var}(X) = p(1-p)$.

Binomial r.v.. with parameter (n,p). Denote $X \sim Bin(n,p)$.

(1) X is the number of successes that occur in the n independent Bernoulli trials with parameter p.

(2) The PMF of X is $p(k) = P\{X=k\} = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0, \dots, n$.

(3) Let $X_1, \dots, X_n$ be independent Bernoulli r.v.s with parameter p. Then $X = \sum_{k=1}^{n} X_k$.

Note $E[X] = np$ and $\text{Var}(X) = np(1-p)$.

Poisson r.v.. with parameter $\lambda > 0$. Denote $X \sim Poisson(\lambda)$.

(1) The PMF of X is $p(k) = P\{X=k\} = \dfrac{e^{-\lambda}\lambda^k}{k!}$ for $k = 0, 1, 2, \dots$

Note $E[X] = \text{Var}(X) = \lambda$.

Geometric random variable & computing examples

Example 1 (Geometric r.v. with parameter p)**.** Denote $X \sim Geom(p)$.

(1) X is the number of independent Bernoulli trials with parameter $p \in (0,1)$ such that first success occur.

(2) By independence, the PMF is $p(k) = (1-p)^{k-1}p$ for $k = 1, 2, 3, \dots$

Then we show $E[X] = \dfrac{1}{p}$ and $\text{Var}(X) = \dfrac{1-p}{p^2}$.

By definition, $E[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p$. Note that for $x \in (-1,1)$, (by a result about the uniform convergence of power series) we have

$$\sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty} (x^k)' = (\sum_{k=0}^{\infty} x^k)' = (\frac{1}{1-x})' = \frac{1}{(1-x)^2}.$$

Then setting $x = 1-p$ leads to $E[X] = p\sum_{k=1}^{\infty} k(1-p)^{k-1} = p\dfrac{1}{[1-(1-p)]^2} = \dfrac{1}{p}$.

To obtain $\text{Var}(X)$, it suffices to compute $E[X^2]$

$$\begin{aligned}
E[X^2] &= \sum_{k=1}^{\infty} k^2(1-p)^{k-1}p \\
&= \sum_{k=1}^{\infty} (k-1+1)^2(1-p)^{k-1}p \\
&= \sum_{k=1}^{\infty} (k-1)^2(1-p)^{k-1}p + \sum_{k=1}^{\infty} 2(k-1)(1-p)^{k-1}p + \sum_{k=1}^{\infty} (1-p)^{k-1}p \\
\text{(let } n = k-1) \quad &= (1-p)\sum_{n=1}^{\infty} n^2(1-p)^{n-1}p + 2(1-p)\sum_{k=1}^{\infty} n(1-p)^{n-1}p + 1 \\
&= (1-p)E[X^2] + 2(1-p)E[X] + 1
\end{aligned}$$

Then substitute $E[X] = \dfrac{1}{p}$ and solve the equation to get $E[X^2] = \dfrac{2-p}{p^2}$.

Hence $\text{Var}(X) = E[X^2] - (E[X])^2 = \dfrac{1-p}{p^2}$.

Remark. Observe that we have use two different ways to compute $E[X]$ and $\text{Var}(X)$ in Example 1 both of which can be recursively extended to deal with series like $\sum_{k=1}^{\infty} k^p x^k$ with $p \in \mathbb{N}$.

Example 2. Let $X \sim Bin(n,p)$. Prove

$$E\left[\frac{1}{1+X}\right] = \frac{1-(1-p)^{n+1}}{(n+1)p}.$$

Proof. Recall the PMF of $Bin(n,p)$ and by the formula of $E[g(X)]$,

$$\begin{aligned}
E\left[\frac{1}{1+X}\right] &= \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} p^k(1-p)^{n-k} \\
&= \frac{1}{(n+1)p}\sum_{k=0}^{n} \frac{1}{k+1}\frac{(n+1)!}{k![(n+1)-(k+1)]!} \cdot p^{k+1}(1-p)^{[(n+1)-(k+1)]} \\
&= \frac{1}{(n+1)p}\sum_{k=0}^{n} \binom{n+1}{k+1} p^{k+1}(1-p)^{[(n+1)-(k+1)]} \\
\text{(let } j = k+1) \quad &= \frac{1}{(n+1)p}\left[\sum_{j=0}^{n} \binom{n+1}{j} p^j(1-p)^{[(n+1)-j]} - (1-p)^{n+1}\right] \\
\text{(by Binomial Thm)} \quad &= \frac{1}{(n+1)p}[1-(1-p)^{n+1}].
\end{aligned}$$

□

Example 3. Let X be a r.v. with non-negative integral values. Prove

$$\sum_{k=0}^{\infty} kP(X \geq k) = \frac{1}{2}(E[X^2] + E[X]).$$

Proof.

$$\begin{aligned}\sum_{k=0}^{\infty} kP(X \geq k) &= \sum_{k=0}^{\infty}\sum_{i=k}^{\infty} kP(X = i) \\ &= \sum_{i=0}^{\infty}\sum_{k=0}^{i} kP(X = i) \\ &= \sum_{i=0}^{\infty} \frac{i(i+1)}{2} P(X = i) \\ &= \frac{1}{2}E[X^2] + \frac{1}{2}E[X],\end{aligned}$$

where in the second equality we have changed the order of summation. □