General information

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- Time and venue: Mon. 11:30AM - 12:15PM, Wu Ho Man Yuen Bldg 407;
- Textbook: Sheldon Ross, *A first course in probability.* 8th edition. Pearson Education International.
- All the suggestions and feedback are welcome.

Basic counting models

Why should we count? First, it gives us the total number of the outcomes of some experiment, i.e., the size of (discrete) sample space, which is the basic of further study. Second, during the process of counting, we can have a better understanding of the considered experiment. Third, countings arise frequently in our daily lives. And etc.

Setup: Last year, we have **47** students in session **A** and **50** students in session **B**.

Example 1. How many possible selections of two students, one from session A and the other from session B, do we have?

Solution. 47×50. Because there are 47×50 vectors (a, b) with $a \in A$ and $b \in B$ and each selection can be represented by such a vector while any two different vectors represent different selections. □

Model 1 (Basic counting principle)**.** *Suppose the experiment A has* **m** *possible outcomes and for each outcome of A there are* **n** *outcomes of experiment B. Then together there are* **mn** *outcomes.*

Note that we can visualized this process in our mind and this principle underlies many counting arguments.

Example 1′. Let $\alpha \in A$ and $\beta \in B$ be two students. What if α, β should be selected together?

Solution. $47 \times 50 - 49 - 46 = 46 \times 49 + 1$. LHS is followed by removing the unsatisfied pairs in which α and β are not matched together. RHS is obtained by first arranging all the other students except α and β, then plus the one pair (α, β). □

In general, more than one reasoning or counting methods can work. We can pick the ones that we are comfortable with or collect them all. Like above, two of the major ways of thinking: (1) follow the description of the experiments, count in the respective cases, and finally sum them up. (2) find a larger set of possible outcomes and then 'remove' the over-counting ones.

Example 2. In a classroom with 47 seats, how many seat plans does session A have?

Solution. $47 \times 46 \times \cdots \times 1 =: 47!$. Determine the possible selections one by one from the first seat to the last seat. □

Model 2 (Permutation)**.** *{# permutations of n elements}* $= n(n-1)\cdots 1 =: n!$ *with convention* $0! = 1$.

Example 2′. What if we only want to know who are sitting in the 1st, 2nd and 3rd seats?

Solution. 'Forward' reasoning: $47 \times 46 \times 45$. The 47 possible students in the 1st seat, multiplies the 46 students in 2nd seat (after fixing the student in 1st seat) and similar for the 3rd seat.

'Exclusion' reasoning; $47!/44!$. We have 47! permutations of 47 students. However, no matter how we permute the 44 students in the last 44 seats, the first three students remain the same to us. Hence dividing the over-counting 44! permutations yields the answer. □

Model 2′. {# ways to choose k **ordered** elements from n elements} $= n(n-1)\cdots(n-(k-1)) = n!/(n-k)!$.

Example 3. What if we just want to know who are sitting in the first three seats and do **not** care the order?

Solution. $(47!/44!)/3!$. Further divide the over-counted 3! permutations of 3 students. □

Model 3 (Combination)**.** *{# ways to choose k elements from n elements}* $= \frac{n!}{k!(n-k)!} =: \binom{n}{k}$.

Example 3′. Suppose the three TAs, denoted by F, X, and W, are supposed to mark the corresponding number of HWs (F: 12, X: 10, and W: 25). How many HW marking plans do the TAs have?

Solution. $\binom{47}{12}\binom{35}{10}\binom{25}{25} = \frac{47!}{12!10!25!}$. Similarly, we can also reason in two ways. □

Model 3′. {# ways to choose $n_1, \ldots, n_k$ elements with $n_1 + \cdots + n_k = n$ into k **distinct** groups from n elements} $= \frac{n!}{n_1!\cdots n_k!}$.

Until now, we have reviewed the basic counting principle, permutations and combinations which are the core concepts in basic counting.