Midterm Examination

1. For $n = 0$,

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ dx = a_0.$$

For $n \geq 1$,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\ dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \left(\cos(nx) - i\sin(nx)\right)\ dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\cos(nx) dx - \frac{i}{2\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)\ dx \\ &= \frac{1}{2}(a_n - ib_n). \end{aligned}$$

Similarly,

$$\begin{aligned} c_{-n} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx}\ dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \left(\cos(nx) + i\sin(nx)\right)\ dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)) dx + \frac{i}{2\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)\ dx \\ &= \frac{1}{2}(a_n + ib_n). \end{aligned}$$

2. (a) For $n = 0$,

$$\begin{aligned} \hat{f}(0) &= \frac{1}{\pi} \int_{0}^{\pi} x\ dx \\ &= \frac{\pi}{2}. \end{aligned}$$

For $n \neq 0$, then using f is even we have

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\ dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| \cos(nx)\ dx \\ &= \frac{1}{\pi} \int_{0}^{\pi} x\cos(nx)\ dx \\ &= \frac{1}{n\pi} \left(- \int_{0}^{\pi} \sin(nx)\ dx \right) \\ &= \frac{(-1)^n - 1}{n^2\pi}. \end{aligned}$$

(b) By best approximation theorem (Lemma 1.2, Chapter 3), when $P(x) = \sum_{|n|\le 2} \hat{f}(n)e^{inx}$, $\|f-P\|$ obtains its minimum. By Pythagorean theorem,

$$\begin{aligned}\|f-P\|^2 &= \|f\|^2 - \|P\|^2 \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx - \sum_{|n|\le 2}\left|\hat{f}(n)\right|^2 \\ &= \frac{\pi^2}{3} - \left(\frac{\pi^2}{4} - \frac{8}{\pi^2}\right) \\ &= \frac{\pi^2}{12} - \frac{8}{\pi^2}.\end{aligned}$$

Hence,

$$\|f-P\| = \sqrt{\frac{\pi^2}{12} - \frac{8}{\pi^2}}.$$

3. (a) For $n = 0$,

$$\begin{aligned}\hat{f}(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\alpha x)\,dx \\ &= \frac{\sin(\pi\alpha)}{\pi\alpha}.\end{aligned}$$

For $n \neq 0$, then using f is even we have

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\alpha x)e^{-inx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\alpha x}+e^{-i\alpha x}}{2}e^{-inx}\,dx \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi}e^{i(\alpha-n)x}\,dx + \frac{1}{4\pi}\int_{-\pi}^{\pi}e^{i(-\alpha-n)x}\,dx \\ &= \frac{1}{2\pi}\left(\frac{\sin(\pi(\alpha-n))}{(\alpha-n)} + \frac{\sin(\pi(\alpha+n))}{(\alpha+n)}\right) \\ &= \frac{(-1)^n\alpha\sin(\pi\alpha)}{\pi(\alpha^2-n^2)}.\end{aligned}$$

Hence,

$$f(x) \sim \frac{\sin(\pi\alpha)}{\pi\alpha} + \sum_{n\neq 0}\frac{(-1)^n\alpha\sin(\pi\alpha)}{\pi(\alpha^2-n^2)}e^{inx}.$$

(b) Since f is a C^1 function and its Fourier Series is absolutely convergent, we have

$$f(x) = \frac{\sin(\pi\alpha)}{\pi\alpha} + \sum_{n\neq 0}\frac{(-1)^n\alpha\sin(\pi\alpha)}{\pi(\alpha^2-n^2)}e^{inx}.$$

Putting $x = \pi$, we get

$$\cos(\pi\alpha) = \frac{\sin(\pi\alpha)}{\pi\alpha} - \sum_{n=1}^{\infty}\frac{2\alpha\sin(\pi\alpha)}{\pi(n^2-\alpha^2)}.$$

Hence,

$$\sum_{n=1}^{\infty}\frac{1}{\pi(n^2-\alpha^2)} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha\tan(\alpha\pi)}.$$

4. (a) If f is an integrable function on $[-\pi, \pi]$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2.$$

(b) If f, g are integrable functions on $[-\pi, \pi]$, so are $f+g, f-g, f+ig, f-ig$. Using the notation of the inner product

$$(f, g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)} \, dx,$$

and the norm

$$\|f\|^2 = (f, f),$$

by Parseval identity and linearity of Fourier operator, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)} \, dx = (f, g) \\
&= \frac{1}{4}\left[\|f+g\|^2 - \|f-g\|^2 + i(\|f+ig\|^2 - \|f-ig\|^2)\right] \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{4}\left[\left|\widehat{(f+g)}(n)\right|^2 - \left|\widehat{(f-g)}(n)\right|^2 + i\left(\left|\widehat{(f+ig)}(n)\right|^2 - \left|\widehat{(f-ig)}(n)\right|^2\right)\right] \\
&= \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}.
\end{aligned}$$

5. Suppose on the contrary there exists $\theta_0 \in [-\pi, \pi]$ such that f is continuous at θ_0 but $f(\theta_0) \neq 0$. Without loss of generality, assume $\theta_0 \in [0, \pi]$. Then there exists $\delta \in (0, \pi)$ such that for any $x \in (\theta_0 - \delta, \theta_0)$, $|f(x)| > \frac{1}{2}|f(\theta_0)| > 0$. By Parseval identity,

$$\begin{aligned}
0 &= \sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx \\
&\geq \frac{1}{2\pi}\int_{\theta_0-\delta}^{\theta_0} |f(x)|^2 \, dx \\
&\geq \frac{\delta}{8\pi}|f(\theta_0)|^2 \\
&> 0.
\end{aligned}$$

This gives a contradiction!

6. (a) Since $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} < \infty$, we have $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} e^{inx}$ is absolutely convergent. Define $f : [-\pi, \pi] \to \mathbb{C}$ by

$$f(x) := \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} e^{inx}.$$

Then $\hat{f}(n) = \frac{1}{n\sqrt{n}}$.

(b) Suppose on the contrary there exists a Riemann integrable function on the circle with the given Fourier series. By Parseval identity,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{1}{n} &= \sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2 \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \ dx \\ &< \infty.\end{aligned}$$

This gives a contradiction!