1.6. Genesis of Fourier analysis.

Now let us consider the wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0 & (1) \\ u(0,t) = u(\pi,t) = 0 & (2) \\ u(x,0) = f(x), \quad \dfrac{\partial}{\partial t} u(x,0) = 0 & (3) \end{cases}$$

(Initial condition)

Recall that we have derived a solution

$$u = \sum_{m=1}^{\infty} A_m \cos mt \sin mx$$

to Eq (1), (2), (3), provided that $(A_m)_{m=1}^{\infty}$ satisfies

$$\sum_{m=1}^{\infty} A_m \sin(mx) = f(x). \quad (*)$$

It now generates a natural question:

Q: For a reasonable function f on $[0, \pi]$, Can we find coefficients A_m such that $(*)$ holds?

This is a fundamental question in Fourier analysis.

Suppose the answer is yes, then formally we have

$$\int_0^{\pi} f(x) \sin nx \, dx = \int_0^{\pi} \left(\sum_{m=1}^{\infty} A_m \sin mx \right) \sin nx \, dx$$

$$= \sum_{m=1}^{\infty} A_m \int_0^{\pi} \sin mx \sin nx \, dx$$

$$= A_n \cdot \frac{\pi}{2},$$

here we use the fact that

$$\int_0^{\pi} \sin nx \sin mx \, dx = \begin{cases} \pi/2 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \cdots$$

Now we call

$$\sum_{m=1}^{\infty} A_n \sin nx$$

the Fourier sine series of f on $[0, \pi]$.

1.7 Fourier series

Suppose a "reasonable" function f on $[0,\pi]$ has a Fourier sine series. Then extend f to an odd function on $[-\pi,\pi]$, f still has the same sine series on $[-\pi,\pi]$.

We may believe that an even function g on $[-\pi,\pi]$ also has a cosine series

$$g(x) = \sum_{m=0}^{\infty} B_m \cos mx.$$

(Formal argument : g' is odd, so

$$g'(x) = \sum_{m=1}^{\infty} B_m \sin mx$$

$$\Rightarrow \quad g(x) = \int g'(x)\,dx = \sum_{m=0}^{\infty} B_m \cos mx \;)$$

Notice that every function F on $[-\pi,\pi]$ can be expressed as $f+g$, where f is odd and g is even.

To see so, let $f(x) = \dfrac{F(x)-F(-x)}{2}$, $g(x) = \dfrac{F(x)+F(-x)}{2}$.

Then we can express that

$$F(x) = \sum_{m=0}^{\infty} a_m \cos(mx) + \sum_{m=1}^{\infty} b_m \sin(mx)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

(using Euler's identity $e^{ix} = \cos x + i \sin x$)

from which $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$, $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

Using the fact $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} e^{imx} e^{-inx} dx = \begin{cases} 1 & \text{if } n=m \\ 0 & \text{otherwise}, \end{cases}$

we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx = C_n .$$

We call C_n the n-th Fourier coefficient of f on $[-\pi, \pi]$

and $\sum_{-\infty}^{\infty} C_n e^{inx}$ the Fourier series of f on $[-\pi, \pi]$.