TA's solution* to 3093 assignment 9

Ch5, Ex11. (5 marks)

To show u is continuous on $\mathbb{R} \times [0, \infty)$, we want to show that

$$\lim_{(x,t)\to(x_0,t_0)} u(x,t) = u(x_0, t_0)$$

for all $(x_0, t_0) \in \mathbb{R} \times [0, \infty)$. As the April 09 lecture note theorem 3 shows†, u is infinitely differentiable on $\mathbb{R} \times (0, \infty)$, therefore it is continuous on $\mathbb{R} \times (0, \infty)$‡. It remains to show that it is continuous on $\mathbb{R} \times \{0\}$.

We know already that (by the good kernel argument)

$$u(x,t) \to f(x) \text{ as } t \downarrow 0, \text{ uniformly in } x.$$

Also, $u(x, 0) = f(x)$ is a continuous function in x. Given $(x,t) \in \mathbb{R} \times [0, \infty)$, $(x_0, 0) \in \mathbb{R} \times \{0\}$, we have

$$\begin{aligned} u(x,t) - u(x_0, 0) &= (u(x,t) - u(x,0)) + (u(x,0) - u(x_0,0)) \\ &= (u(x,t) - f(x)) + (f(x) - f(x_0)). \end{aligned}$$

It follows that u is continuous on $\mathbb{R} \times \{0\}$.

To see it vanishes at infinity, we note the following two estimates:

$$|u(x,t)| \leq \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} |f(x-y)| dy \leq \frac{c}{\sqrt{t}},$$

and

$$\begin{aligned} |u(x,t)| &\leq \int_{|y|\leq|x|/2} |f(x-y)|\mathcal{H}_t(y)dy + \int_{|y|>|x|/2} |f(x-y)|\mathcal{H}_t(y)dy \\ &\leq \frac{C}{1+|x|^2} + Ct^{-1/2}e^{-cx^2/t}. \end{aligned}$$

Here, f is rapidly decreasing, so $|f(x-y)| \leq C/(1+|x|^2)$ on $|y| \leq |x|/2$; and $\mathcal{H}_t(y) \leq Ct^{-1/2}e^{-cx^2/t}$ if $|y| > |x|/2$. To obtain it vanishes at infinity as $|x| + t \to \infty$, we note that if $|x| \leq t$, then $t \to \infty$. We have

$$|u(x,t)| \leq \frac{C}{\sqrt{t}} \to 0.$$

On the other hand, if $|x| > t$, then $|x| \to \infty$, we have

$$|u(x,t)| \leq \frac{C}{1+|x|^2} + Ct^{-1/2}e^{-cx^2/t} \to 0.$$

*This solution is adapted from the work by former TAs.

†Thanks to a student for citing this reference.

‡Be aware that a function $g : \mathbb{R}^n \to \mathbb{R}$ that has first-order partial derivatives need not be continuous. But if it is continuously differentiable then it is continuous. C.f. Ch13 of Fitzpatrick's Advanced Calculus (white and blue cover).

Alternative for the continuity part[§]*:*

From the textbook, we know that

$$u(x,t) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{-4\pi^2 t\xi^2} e^{2\pi i\xi x} d\xi.$$

As $f \in \mathcal{S}(\mathbb{R})$, the Fourier inversion formula holds for f, whence the above also holds when $t = 0$. We want to show that

$$\lim_{(x,t)\to(x_0,t_0)} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{-4\pi^2 t\xi^2} e^{2\pi i\xi x} d\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{-4\pi^2 t_0\xi^2} e^{2\pi i\xi x_0} d\xi.$$

This can follow from the Lebesgue dominated convergence theorem with the dominator function $g(\xi) := \left|\widehat{f}(\xi)\right|$.

Ex12. (5 marks) (It seems that you already have good solution from tutorial??)

We make two remarks. Firstly, for the heat equation part, one may also try to verify and make use of the following equation:

$$\frac{\partial \mathcal{H}_t(x)}{\partial t} = \frac{\partial^2 \mathcal{H}_t(x)}{\partial x^2}.$$

Secondly, when considering $\dfrac{x^2}{4t} \equiv c$, by $t > 0$ we implicitly require that $c > 0$. On the other hand, this relation does not automatically give $x = \sqrt{4tc}$, because x may be negative.

[§]A student provides this solution.