TA's solution*† to 3093 assignment 8

Ch5, Ex3. (3 marks)

(a) We follow the hint. Since there exist $A_1, A_2 > 1$ s.t. $\left|\widehat{f}(\xi)\right| \le \frac{A_1}{|\xi|^{1+\alpha}}$ for all $|\xi| > A_2$, we have

$$\left|\widehat{f}(\xi)\right| \le \frac{\left(1 + A_2^{1+\alpha}\right) \cdot \left(\sup_{t\in[-A_2,A_2]} \left|\widehat{f}(t)\right|\right)}{1 + |\xi|^{1+\alpha}} + \frac{2A_1}{1 + |\xi|^{1+\alpha}}$$

for all $\xi \in \mathbb{R}$.‡ Therefore, the Fourier inversion formula holds. It follows that for $h \neq 0$,

$$\begin{aligned}
|f(x+h) - f(x)| &= \left|\int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2i\pi x\xi} \left(e^{2i\pi h\xi} - 1\right) d\xi\right| \\
&\ll \int_{-1/|h|}^{1/|h|} \left|\widehat{f}(\xi) \left(e^{2i\pi h\xi} - 1\right)\right| d\xi + \int_{\mathbb{R}\setminus[-1/|h|,1/|h|]} \left|\widehat{f}(\xi)\right| d\xi \\
&\ll \int_{-1/|h|}^{1/|h|} \left|\widehat{f}(\xi)\right| |h\xi| \, d\xi + \int_{1/|h|}^{\infty} \frac{1}{\xi^{1+\alpha}} d\xi \\
&\ll \int_{0}^{1/|h|} \frac{|h| \, \xi}{\xi^{1+\alpha}} d\xi + \int_{1/|h|}^{\infty} \frac{1}{\xi^{1+\alpha}} d\xi \\
&\ll |h|^{\alpha}.
\end{aligned}$$

Else if $h = 0$, then the above holds plainly. Done.

(b) It is proved by contradiction. If $\widehat{f}$ is not continuous, then by definition it is not in $\mathcal{M}(\mathbb{R})$. Else, $\widehat{f}$ is continuous. Suppose there exists $\varepsilon \in (0, 1)$ s.t. $\left|\widehat{f}(\xi)\right| \ll \frac{1}{|\xi|^{1+\varepsilon}}$ for all large ξ. Then by the result of part (a), for all small $h \neq 0$ we have

$$\left|\frac{1}{\log(1/|h|)}\right| = |f(h) - f(0)| \ll |h|^{\varepsilon}.$$

This is a contradiction however, because when $h \to 0$, we have by L'Hospital's Rule

$$\left|\frac{|h|^{-\varepsilon}}{\log(1/|h|)}\right| \to \infty.$$

Since $\frac{1}{1 + |\xi|^2} \le \frac{1}{|\xi|^{1+\varepsilon}}$ for all $|\xi| \ge 1$, it follows that $\widehat{f}$ is not in $\mathcal{M}(\mathbb{R})$.

Ex5. (3 marks)

(a) We first show that $\widehat{f}$ is continuous. Fix an $\varepsilon > 0$. Since f is of moderate decrease, there is an $A > 0$ s.t.

$$\int_{\mathbb{R}\setminus[-A,A]} |f(x)| \, dx < \varepsilon.$$

*This solution is adapted from the work by former TAs.

†In this solution we make use of the Vinogradov "$\ll$" notation introduced in the TA's solution to assignment 5.

‡The first term on the R.H.S. handles the case $\xi \in [-A_2, A_2]$. The second term handles the case $\xi \in \mathbb{R} \setminus [-A_2, A_2]$.

And for this fixed A, there is an $\delta > 0$ s.t. for all h with $|h| < \delta$, we have

$$\int_{[-A,A]} \left|f(x)\left(e^{-2i\pi xh}-1\right)\right| dx \ll \int_{[-A,A]} |f(x)xh|\, dx = |h| \int_{[-A,A]} |f(x)x|\, dx < \varepsilon.$$

As a result, independent of ξ, there is an $\delta > 0$ s.t. for all h with $|h| < \delta$, we have

$$\begin{aligned}\left|\widehat{f}(\xi+h)-\widehat{f}(\xi)\right| &= \left|\int_{-\infty}^{\infty} f(x)e^{-2i\pi x\xi}\left(e^{-2i\pi xh}-1\right) dx\right| \\ &\ll \int_{\mathbb{R}\setminus[-A,A]} |f(x)|\, dx + \int_{[-A,A]} \left|f(x)\left(e^{-2i\pi xh}-1\right)\right| dx \ll \varepsilon,\end{aligned}$$

which was to be demonstrated.

Next we show that $\widehat{f}(\xi) \to 0$ when $|\xi| \to \infty$. For $\xi \neq 0$ we have

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2i\pi x\xi} dx = \int_{-\infty}^{\infty} f(x-\frac{1}{2\xi})e^{-2i\pi(x-\frac{1}{2\xi})\xi} dx = e^{i\pi}\int_{-\infty}^{\infty} f(x-\frac{1}{2\xi})e^{-2i\pi x\xi} dx,$$

whence

$$\widehat{f}(\xi) = \frac{1}{2}\int_{-\infty}^{\infty}\left(f(x)-f(x-\frac{1}{2\xi})\right)e^{-2i\pi x\xi} dx.$$

Fix an $\varepsilon > 0$. Since f is of moderate decrease, there is an $A > 10$ s.t.

$$\int_{\mathbb{R}\setminus[-A+1,A-1]} |f(x)|\, dx < \varepsilon.$$

It follows that for any $|\xi| > 1$, we have

$$\int_{\mathbb{R}\setminus[-A,A]} \left|f(x-\frac{1}{2\xi})\right| dx < \varepsilon$$

and so

$$\int_{\mathbb{R}\setminus[-A,A]} \left|f(x)-f(x-\frac{1}{2\xi})\right| dx \le \int_{\mathbb{R}\setminus[-A,A]} |f(x)| + \left|f(x-\frac{1}{2\xi})\right| dx \ll \varepsilon.$$

For this fixed A, since f is continuous on $[-A-1, A+1]$, f is uniformly continuous on $[-A-1, A+1]$. Therefore, there exists $R > 0$ s.t. for all ξ with $|\xi| > R$ and for all $x \in [-A, A]$, we have

$$\left|f(x)-f(x-\frac{1}{2\xi})\right| < \frac{\varepsilon}{A}.$$

Combining the results, we see that for all ξ with $|\xi| > R$, we have

$$\begin{aligned}\left|\widehat{f}(\xi)\right| &= \left|\frac{1}{2}\int_{-\infty}^{\infty}\left(f(x)-f(x-\frac{1}{2\xi})\right)e^{-2i\pi x\xi} dx\right| \\ &\ll \int_{\mathbb{R}\setminus[-A,A]} \left|f(x)-f(x-\frac{1}{2\xi})\right| dx + \int_{[-A,A]} \left|f(x)-f(x-\frac{1}{2\xi})\right| dx \ll \varepsilon + A\cdot\frac{\varepsilon}{A} \ll \varepsilon.\end{aligned}$$

Done.[§]

[§]Some students do this question by using Lebesgue dominated convergence theorem which is also good.

(b) Suppose we have the following result:

If f and $\widehat{f}$ are in $\mathcal{M}(\mathbb{R})$ then $f(x) = \int \widehat{f}(\xi)e^{2i\pi\xi x}d\xi$ for all $x \in \mathbb{R}$.

Then the question is solved immediately by noting that the zero function is in $\mathcal{M}(\mathbb{R})$. (It seems that you have already learnt the aforementioned result from the lecture note??)

Ex9. (4 marks) We have

$$\begin{aligned}\int_{-R}^{R}\left(1-\frac{|\xi|}{R}\right)\widehat{f}(\xi)e^{2i\pi x\xi}d\xi &= \int_{-R}^{R}\left(1-\frac{|\xi|}{R}\right)\int_{-\infty}^{\infty}f(t)e^{-2i\pi\xi t}dt\; e^{2i\pi x\xi}d\xi \\ &=^{¶} \int_{-\infty}^{\infty}f(t)\int_{-R}^{R}\left(1-\frac{|\xi|}{R}\right)e^{-2i\pi\xi(t-x)}d\xi dt \\ &= \int_{-\infty}^{\infty}f(t)\widehat{g}(t-x)dt,\end{aligned}$$

where

$$g(y) = g_R(y) := \begin{cases}\left(1-\frac{|y|}{R}\right) & \text{if } |y| \le R \\ 0 & \text{otherwise.}\end{cases}$$

We find that for $\beta \neq 0$,

$$\begin{aligned}\widehat{g}(\beta) &= \int_{-R}^{R}\left(1-\frac{|y|}{R}\right)e^{-2i\pi y\beta}dy = \int_{0}^{R}\left(1-\frac{y}{R}\right)e^{-2i\pi y\beta}dy + \int_{-R}^{0}\left(1-\frac{-y}{R}\right)e^{-2i\pi y\beta}dy \\ &= \int_{0}^{R}\left(1-\frac{y}{R}\right)e^{-2i\pi y\beta}dy + \int_{0}^{R}\left(1-\frac{z}{R}\right)e^{2i\pi z\beta}dz \\ &= 2\int_{0}^{R}\left(1-\frac{y}{R}\right)\cos(2\pi y\beta)dy \\ &= \frac{1}{\pi\beta}\int_{0}^{R}\left(1-\frac{y}{R}\right)d\sin(2\pi y\beta) = \frac{1}{\pi\beta R}\int_{0}^{R}\sin(2\pi y\beta)dy \\ &= \frac{1-\cos(2\pi R\beta)}{\pi\beta R(2\pi\beta)} = \frac{\sin^2(\pi R\beta)}{(\pi\beta)^2 R}.\end{aligned}$$

By Ex5(a), we know that $\widehat{g}$ is continuous, whence by L'Hospital's Rule $\widehat{g}(0) = \lim_{\beta\to 0}\widehat{g}(\beta) = \lim_{\beta\to 0}\dfrac{\pi R\sin(2\pi R\beta)}{2\beta\pi^2 R} = R$. This shows that $\widehat{g} = \mathcal{F}_R$. Back to the beginning equation and noting that $\mathcal{F}_R$ is an even function, we get

$$\int_{-R}^{R}\left(1-\frac{|\xi|}{R}\right)\widehat{f}(\xi)e^{2i\pi x\xi}d\xi = \int_{-\infty}^{\infty}f(t)\mathcal{F}_R(x-t)dt = (f * \mathcal{F}_R)(x).$$

To finish this question, we check that $\{\mathcal{F}_R\}$ is a family of good kernels as $R \to \infty$.

¶By Fubini theorem since (c.f. lecture note) $\int_{-R}^{R}\int_{-\infty}^{\infty}\left|f(t)\left(1-\frac{|\xi|}{R}\right)e^{-2i\pi\xi(t-x)}\right|dtd\xi \ll_R 1$.

- Since both g and $\widehat{g} = \mathcal{F}_R$ are in $\mathcal{M}(\mathbb{R})$ (we have showed that $\mathcal{F}_R$ is continuous at $t = 0$), it follows that
$$1 = g(0) = \int_{-\infty}^{\infty} \widehat{g}(\beta) e^{2i\pi 0 \beta} d\beta = \int_{-\infty}^{\infty} \mathcal{F}_R(\beta) d\beta.$$
- We note that $\mathcal{F}_R(\beta) \geq 0$ for all β.
- Given $\delta > 0$, we have
$$\int_{\mathbb{R}\setminus[\delta,\delta]} \mathcal{F}_R(\beta) d\beta \ll \int_{\delta}^{\infty} R \frac{\left|\sin^2(\pi R \beta)\right|}{(\pi \beta R)^2} d\beta \ll \frac{1}{R} \int_{\delta}^{\infty} \frac{1}{\beta^2} d\beta \ll_{\delta} \frac{1}{R} \to 0 \text{ as } R \to \infty.$$

Done.