TA's solution* to 3093 assignment 4

Ch2, Ex12. (2 marks)†

12 (p.62) We need to prove

$$\frac{s_1 + s_2 + \cdots + s_n}{n} - s = \frac{(s_1 - s) + (s_2 - s) + \cdots + (s_n - s)}{n} \to 0$$

as $n \to \infty$. By replacing s_n with $s_n - s$, it suffices to prove the case for $s = 0$. Now, given any $\epsilon > 0$, we note that $s_n \to 0$ and therefore there exists $N \in \mathbb{N}$ such that $|s_n| < \epsilon$ for all $n > N$. Then we have

$$\begin{aligned}
\left|\frac{s_1 + s_2 + \cdots + s_n}{n}\right| &= \left|\frac{s_1 + \cdots + s_N + s_{N+1} + \cdots + s_n}{n}\right| \\
&\leq \frac{|s_1 + \cdots + s_N|}{n} + \frac{1}{n}(|s_{N+1}| + \cdots + |s_n|) \\
&\leq \frac{|s_1 + \cdots + s_N|}{n} + (\frac{n-N}{n})\epsilon \\
&< \frac{|s_1 + \cdots + s_N|}{n} + \epsilon.
\end{aligned}$$

Since N is fixed and $|s_1 + \cdots + s_N|$ is a finite number, we can choose an integer $N_1 > N$ such that $\frac{|s_1+\cdots+s_N|}{n} < \epsilon$. Hence, whenever $n > N_1$.

$$\left|\frac{s_1 + s_2 + \cdots + s_n}{n}\right| < 2\epsilon.$$

Thus $\sum c_n$ is Cesàro summable to s.

*This solution is adapted from the work by former TAs.

†In this course, most often we deal with complex numbers rather than real numbers. As $\mathbb{C}$ has no natural ordering, inequality is only meaningful when the involved complex numbers are inside absolute values. Sorry that I have also made such mistake in my solution to Hw3 Ex17a ("$\sup f$" should be written as "$\sup |f|$" there).

Ex13. (a)

13(a) (p.62) By letting $c_1' = c_1 - s$, $c_n' = c_n$ for $n \geq 2$, we see that the series $\sum c_n$ is Abel summable to s if and only if c_n' is Abel summable to 0. Hence it suffices to consider $s = 0$. Let $s_0 = 0$ and $s_n = c_1 + ... + c_n$, then

$$\sum_{n=1}^{N} c_n r^n = \sum_{n=1}^{N} (s_n - s_{n-1}) r^n = \sum_{n=1}^{N} s_n r^n - r \sum_{n=1}^{N-1} s_n r^n = (1-r) \sum_{n=1}^{N} s_n r^n + s_N r^{N+1}.$$

Since $s_N r^{N+1} \to 0$ as $N \to \infty$, thus

$$\sum_{n=1}^{\infty} c_n r^n = (1-r) \sum_{n=1}^{\infty} s_n r^n.$$

For any $\epsilon > 0$, by noting that $s_n \to 0$, we can therefore find $N_0 \in \mathbb{N}$, such that $|s_n| < \epsilon$ for $n > N_0$. Moreover, $s_n \to 0$ implies $|s_n| \leq M$, we can find $\delta > 0$, such that $(1-r) M N_0 < \epsilon$ whenever $1 - \delta < r < 1$. Then we have

$$\begin{aligned}\left|(1-r)\sum_{n=1}^{\infty} s_n r^n\right| \leq & \left|(1-r)\sum_{n=1}^{N_0} s_n r^n\right| + (1-r) \sum_{n=N_0+1}^{\infty} |s_n| r^n \\ \leq & (1-r) M N_0 + \epsilon \sum_{n=N_0+1}^{\infty} r^n \\ = & \epsilon + \epsilon = 2\epsilon.\end{aligned}$$

This means $\lim_{r \to 1} (1-r) \sum_{n=1}^{\infty} s_n r^n = 0$. Hence $\sum c_n$ is Abel summable.

(b)

13(b) Let $c_n = (-1)^n$, then $\sum_1^{\infty} (-1)^n$ does not converge. However,

$$\lim_{r \to 1} \sum_{n=1}^{\infty} (-1)^n r^n = \lim_{r \to 1} \frac{-r}{1+r} = -\frac{1}{2}.$$

(c). (2 marks)

Firstly, let's suppose $\lim_{n \to \infty} \sigma_n = 0$, where $\sigma_n := \frac{1}{n} \sum_1^n s_i$. By following the computation in part (a), we have for each $r \in [0, 1)$

$$\sum_{n=1}^{N} c_n r^n = (1-r) \sum_{n=1}^{N} s_n r^n + s_N r^{N+1}.$$

The R.H.S. is

$$\begin{aligned}
&= (1-r)\sum_{n=1}^{N}(n\sigma_n-(n-1)\sigma_{n-1})r^n+(N\sigma_N-(N-1)\sigma_{N-1})r^{N+1}\\
&= (1-r)\left[\sum_{n=1}^{N-1} n\sigma_n r^n-\sum_{n=1}^{N-1} n\sigma_n r^{n+1}+N\sigma_N r^N\right]+(N\sigma_N-(N-1)\sigma_{N-1})r^{N+1}\\
&= (1-r)^2\sum_{n=1}^{N-1} n\sigma_n r^n+N\sigma_N r^N-(N-1)\sigma_{N-1}r^{N+1}.
\end{aligned}$$

As $\{\sigma_n\}$ is bounded, $\lim_{n\to\infty} nr^n=0$, and $\sum nr^n$ converges, we see that it converges to

$$(1-r)^2\sum_{n=1}^{\infty} n\sigma_n r^n$$

when N tends to infinity. This show $\sum c_n r^n$ converges and

$$\sum_{n=1}^{\infty} c_n r^n=(1-r)^2\sum_{n=1}^{\infty} n\sigma_n r^n.$$

Our aim is to show

$$\lim_{r\uparrow 1}\sum_{n=1}^{\infty} c_n r^n=0.$$

Since $\sigma_n\to 0$, there exists $M>0$ such that $|\sigma_n|<M$ for all n. Moreover, given $\varepsilon>0$, there exists N_0 such that $|\sigma_n|<\varepsilon$ for all $n\geq N_0$. Noting the identity

$$\sum_{n=1}^{\infty} nr^n=r\sum_{n=1}^{\infty}\frac{dr^n}{dr}=r\frac{d}{dr}\left(\frac{r}{1-r}\right)=\frac{r}{(1-r)^2},$$

we have

$$(1-r)^2\left|\sum_{N_0+1}^{\infty} n\sigma_n r^n\right|\leq(1-r)^2\left(\sum_{n=1}^{\infty} nr^n\right)\varepsilon=\varepsilon r<\varepsilon.$$

Take $\delta:=\sqrt{\dfrac{\varepsilon}{\left(M\sum_1^{N_0} n\right)}}>0$. We see that whenever $1-\delta<r<1$,

$$\left|\sum_{n=1}^{\infty} c_n r^n\right|\leq\delta^2\left|\left(\sum_{1}^{N_0} n\sigma_n r^n\right)\right|+(1-r)^2\left|\sum_{N_0+1}^{\infty} n\sigma r^n\right|\leq 2\varepsilon.$$

This completes the proof for the case $\lim_{n\to\infty}\sigma_n=0$.
For the general case, suppose $\lim_{n\to\infty}\sigma_n=\sigma$. Define $\widetilde{c_1}:=c_1-\sigma$ and $\widetilde{c_n}:=c_n$ for all $n\geq 2$. Then we have $\widetilde{s_n}=s_n-\sigma$ and $\widetilde{\sigma_n}=\dfrac{\widetilde{s_1}+\cdots+\widetilde{s_n}}{n}=\dfrac{s_1+\cdots+s_n-n\sigma}{n}=\sigma_n-\sigma\to 0$. Therefore, by the above result

$$\lim_{r\uparrow 1}\sum_{n=1}^{\infty}\widetilde{c_n}r^n=0.$$

This implies

$$\lim_{r\uparrow 1}\sum_{n=1}^{\infty} c_n r^n = \lim_{r\uparrow 1}\left(\sigma r + \sum_{n=1}^{\infty} \widetilde{c_n} r^n\right) = \sigma,$$

which was to be demonstrated.

Remark:

To show

$$\sum_{n=1}^{\infty} c_n r^n = (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n,$$

the following argument is not without pain:

> from 13(a),
> we obtain (using the same identity with c_n replaced by s_n)
>
> $$\sum_{n=1}^{\infty} c_n r^n = (1-r)\sum_{n=1}^{\infty} s_n r^n = (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n.$$

It is because the hypothesis in part(a) and part(c) are different. For example, we have to justify why $s_N r^{N+1} \to 0$ when $N \to \infty$ if $\sum c_n$ is merely Cesàro summable.

(d)

> **13(d)** Note that if c_n is Cesàro summable (i.e. $\sigma_n = \frac{s_1 + .. + s_n}{n} \to \sigma$), then
>
> $$\frac{s_n}{n} = \frac{n\sigma_n - (n-1)\sigma_{n-1}}{n} = \sigma_n - \frac{n-1}{n}\sigma_{n-1} \longrightarrow 0.$$
>
> Hence,
>
> $$\frac{c_n}{n} = \frac{s_n - s_{n-1}}{n} = \frac{s_n}{n} - \frac{n-1}{n}\frac{s_{n-1}}{n-1} \longrightarrow 0.$$
>
> If $c_n = (-1)^{n-1}n$, then $\frac{c_n}{n} = (-1)^{n-1}$, which does not converge. Hence, c_n is not Cesàro summable. However, $\sum_{n=1}^{\infty}(-1)^{n-1}nr^n = \frac{r}{(1+r)^2}$, so
>
> $$\lim_{r\to 1}\sum_{n=1}^{\infty}(-1)^{n-1}nr^n = \frac{1}{4}.$$
>
> Hence, it is Abel summable.

Ch3, Ex6. (3 marks)[‡]

Ex 6 (p. 89). Assume that $\{a_k\}$ is the coefficient of some Riemann integrable function f, i.e. $f(x) \sim \sum_{k=1}^{\infty} \frac{e^{ikx}}{k}$. Consider $A_r(f)(0)$,

$$A_r(f)(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) P_r(-\theta) d\theta.$$

$P_r(\theta)$ is an even function on θ, and since $1-2r\cos\theta+r^2 = (1-r\cos\theta)^2+r^2\sin^2\theta > 0$ for $r \in [0,1)$, so $P_r(\theta) = P_r(-\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2} > 0$. We now have

$$\begin{aligned} |A_r(f)(0)| \leq & \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)| P_r(-\theta) d\theta \\ \leq & \frac{1}{2\pi} \sup_\theta |f(\theta)| \int_{-\pi}^{\pi} P_r(-\theta) d\theta \\ = & \sup_\theta |f(\theta)| < \infty. \end{aligned}$$

On the other hand, $\lim_{r\to 1} A_r(f)(0) = \lim_{r\to 1} \sum_{k=1}^{\infty} \frac{r^k}{k} = \infty$. Therefore, there's no function with $\{a_k\}$ as its coefficient.
Note that $\lim_{r\to 1} \sum_{k=1}^{\infty} \frac{r^k}{k} = \infty$ because for all $M > 0$, we can choose N such that $\sum_{k=1}^{N} \frac{1}{k} > 2M$. Then we choose r so close to 1 that $r^N \geq 1/2$, then

$$\sum_{k=1}^{\infty} \frac{r^k}{k} \geq \sum_{k=1}^{N} \frac{r^k}{k} \geq \frac{1}{2} \sum_{k=1}^{N} \frac{1}{k} \geq M.$$

Alternative:

We can also use the Fejér kernel in place of the Poisson kernel. On the one hand, we have $|\sigma_N(f)(0)| = |(f * F_N)(0)| \leq \sup |f|$; while on the other hand

$$\begin{aligned} |\sigma_N(f)(0)| &= \left| \frac{1\cdot(N-1) + \frac{1}{2}\cdot(N-2) + \cdots + \frac{1}{N-1}\cdot(1)}{N} \right| \\ &\geq \frac{1\cdot(N-1) + \frac{1}{2}\cdot(N-2) + \cdots + \frac{1}{\lfloor N/2 \rfloor}\cdot(N - \lfloor N/2 \rfloor)}{N}, \end{aligned}$$

which is $\gtrsim \frac{1}{2}(1 + \frac{1}{2} + \cdots + \frac{1}{N/2}) \to \infty$ when $N \to \infty$.

Ex8. (a). (1 marks)

Ex 8a (p. 89). $\widehat{f}(n)$ is the same as that of Exercise 6 in Chapter 2. Using the Parseval's identity:

$$(\frac{\pi}{2})^2 + 2\sum_{n=0}^{\infty} \left(\frac{-2}{(2n+1)^2\pi} \right)^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta = \frac{\pi^2}{3}.$$

This implies

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

Also, $\sum_{n=1}^{\infty} \frac{1}{n^4} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} + \frac{1}{2^4} \sum_{n=1}^{\infty} \frac{1}{n^4}$. Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

[‡]Please refer to textbook Ch3 Sec2.2 p.83-84 for more explanation of this question. Thanks to a student for citing this reference.

(b)

Ex 8b. We have computed $\widehat{f}(n)$ in Exercise 4 in Chapter 2. Using the same method as (a), we have

$$2 \cdot \frac{16}{\pi^2} \cdot \sum_{k>0, k \text{ odd}} \frac{1}{k^6} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta = \frac{\pi^4}{30}.$$

Hence, $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960}$ follows. As

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} + \frac{1}{2^6} \sum_{n=1}^{\infty} \frac{1}{n^6},$$

we have $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$.

Ex9. (2 marks) Note that

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\pi}{\sin \pi\alpha} e^{i(\pi - x)\alpha} e^{-inx} dx = \frac{1}{2 \sin \pi\alpha} \int_0^{2\pi} e^{i\pi\alpha} e^{-i(n+\alpha)x} dx$$

$$= \frac{e^{i\pi\alpha}}{2 \sin \pi\alpha} \left[-\frac{e^{-i(n+\alpha)x}}{i(n+\alpha)} \right]_0^{2\pi} = \frac{e^{i\pi\alpha}}{2 \sin \pi\alpha} \frac{1 - e^{-i\alpha 2\pi}}{i(n+\alpha)} = \frac{1}{n+\alpha}.$$

Hence the Fourier series of f is $\sum_{n=-\infty}^{\infty} \dfrac{e^{inx}}{n+\alpha}$. By the Parseval's identity, we get

$$\sum_{-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\pi}{\sin \pi\alpha} e^{i(\pi - x)\alpha} \right|^2 dx = \frac{\pi^2}{\sin^2 \pi\alpha}.$$