TA's solution* to 3093 assignment 3

Ch2, Ex9. (6 marks)

9(a) (p.61) It is easy to see $\widehat{f}(0) = \frac{b-a}{2\pi}$. If $n \neq 0$,

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_a^b e^{-inx}dx = \frac{e^{-ina}-e^{-inb}}{2\pi in}.$$

Hence,

$$f(x) \sim \frac{b-a}{2\pi} + \sum_{n\neq 0} \frac{e^{-ina}-e^{-inb}}{2\pi in} e^{inx}.$$

(b) We want to show that if $b-a \neq 0, 2\pi$, then

$$\sum_{n\neq 0} \left|\frac{e^{-ina}-e^{-inb}}{n}\right| = \infty.$$

This can be done by one of the following arguments:

Approach 1

Let $\phi_0 := b-a$. We have

$$\left|\frac{e^{-ina}-e^{-inb}}{n}\right| = \left|\frac{e^{in(b-a)}-1}{n}\right| = \left|\frac{e^{in\phi_0}-1}{n}\right|.$$

Observe that the sequence $\{z_n\} := \left\{e^{in\phi_0}\right\}_{n=1}^{\infty}$ represents successive rotations on the unit circle $\mathbb{T}$ in the complex plane by angle ϕ_0 ($z_{n+1} = z_n \cdot e^{i\phi_0}$). We want to show that for many values of n, $\left|e^{in\phi_0}-1\right|$ is not small. Since $0 < \phi_0 < 2\pi$, we claim that if z_n is close to 1, then after rotating it by angle ϕ_0, its new position (i.e. z_{n+1}) is no longer close to 1. It should be clear by drawing a picture. To be precise,

$$\begin{cases} k_0 \in \mathbb{Z} \\ |n\phi_0 - 2k_0\pi| \leq \dfrac{\min\{\phi_0, 2\pi-\phi_0\}}{2} := \varepsilon_0 \end{cases}$$

$$\Rightarrow \begin{cases} (n+1)\phi_0 - 2k_0\pi \geq \phi_0 - |n\phi_0 - 2k_0\pi| \geq \phi_0 - \dfrac{\phi_0}{2} \geq \varepsilon_0 \\ 2(k_0+1)\pi - (n+1)\phi_0 = (2\pi-\phi_0) + (2k_0\pi - n\phi_0) \geq (2\pi-\phi_0) - \dfrac{(2\pi-\phi_0)}{2} \geq \varepsilon_0. \end{cases}$$

As a result, for $c := |e^{i\varepsilon_0}-1| > 0$, we have $|z_n - 1| < c \Rightarrow |z_{n+1}-1| \geq c$, whence

$$\sum_{n\neq 0}\left|\frac{e^{-ina}-e^{-inb}}{n}\right| \geq \sum_{n=1}^{\infty}\left|\frac{e^{-ina}-e^{-inb}}{n}\right| = \sum_{n=1}^{\infty}\left|\frac{z_n-1}{n}\right| \geq^{\dagger} \sum_{\substack{n\in\mathbb{N}\\ n \text{ even}}} \frac{c}{n} = \infty.$$

*This solution is adapted from the work by former TAs.

†Let $\{i_1, i_2, \ldots\}$ be precisely those positive integers such that $|z_{i_k} - 1| \geq c$. Then $2k \geq i_k$ by induction on k: if $i_{k+1} \neq i_k + 1$, then $i_{k+1} = i_k + 2 \leq 2k + 2 = 2(k+1)$; else $i_{k+1} = i_k + 1 \leq 2k + 1 \leq 2(k+1)$.

Approach 2

9(b). The Fourier series does not converge absolutely means that we need to show

$$\sum_{n\neq 0}\left|\frac{e^{-ina}-e^{-inb}}{2\pi in}e^{inx}\right| = \infty.$$

Denote $\theta_0 = \frac{b-a}{2}$,

$$\begin{aligned}\sum_{n\neq 0}\left|\frac{e^{-ina}-e^{-inb}}{2\pi in}e^{inx}\right| &= \sum_{n\neq 0}\left|e^{-in(b+a)/2}\,\frac{e^{in(b-a)/2}-e^{-in(b-a)/2}}{2\pi in}\right| \\ &= \sum_{n\neq 0}\left|\frac{\sin n\theta_0}{\pi n}\right|.\end{aligned} \tag{0.1}$$

Note that from the assumption, $\theta_0 < \pi$. Hence, we can find some $c > 0$ so that

$$\frac{\pi - 2\sin^{-1} c}{\theta_0} > 1.$$

This means for all integers $k \geq 1$, the length of the intervals $(\frac{\pi k+\sin^{-1} c}{\theta_0}, \frac{\pi(k+1)-\sin^{-1} c}{\theta_0})$ is equal to $\frac{\pi-2\sin^{-1} c}{\theta_0} > 1$. This implies there exists some integer n_k such that

$$n_k\theta_0 \in (\pi k + \sin^{-1} c, \pi(k+1) - \sin^{-1} c).$$

This means that $n_k \leq \frac{\pi(k+1)-\sin^{-1} c}{\theta_0} \leq \frac{\pi(k+1)}{\theta_0}$ and $|\sin n_k\theta_0| \geq c$. Hence,

$$(0.1) \geq \sum_{n>0}\left|\frac{\sin n\theta_0}{\pi n}\right| \geq \sum_{k=1}^{\infty}\left|\frac{\sin n_k\theta_0}{\pi n_k}\right| \geq \sum_{k=1}^{\infty}\frac{c}{\pi\frac{\pi(k+1)}{\theta_0}} = \frac{\theta_0}{c\pi^2}\sum_{k=1}^{\infty}\frac{1}{k+1}.$$

As the series $\sum_{k=1}^{\infty}\frac{1}{k+1} = \infty$, the proof is completed.
Remark. *Drawing a graph of* $y = |\sin x|$ *helps visualizing the argument.*

Approach 3

This approach combines the hint of the question with the observation in approach 1. The key point is that if $|\sin n\theta_0| \approx 0$, then $|\cos n\theta_0| \approx 1$, so $|\sin(n+1)\theta_0| = |\sin n\theta_0\cos\theta_0 + \cos n\theta_0 \sin\theta_0| \geq |\cos n\theta_0 \sin\theta_0| - |\sin n\theta_0\cos\theta_0| \approx |\sin\theta_0|$. We omit the detail.

Approach 4[‡]

Observe that

$$\sum_{n\geq 1}\left|\frac{\sin n\theta_0}{n}\right| \geq \sum_{n\geq 1}\frac{\sin^2 n\theta_0}{n} = \sum_{n\geq 1}\frac{1-\cos 2n\theta_0}{2n}.$$

By Dirichlet's test $\sum_{n\geq 1}\dfrac{\cos 2n\theta_0}{n}$ converges (see part (c)), while $\sum\dfrac{1}{2n} = \infty$. The result follows.

[‡]A student provides this solution.

(c)

9(c). Note that

$$\begin{aligned} & \frac{e^{-ina}-e^{-inb}}{2\pi in}e^{inx}+\frac{e^{-i(-n)a}-e^{-i(-n)b}}{2\pi i(-n)}e^{i(-n)x} \\ = & \frac{1}{2\pi in}\left[(e^{in(x-a)}-e^{-in(x-a)})-(e^{in(x-b)}-e^{-in(x-b)})\right] \\ = & \frac{1}{\pi n}(\sin n(x-a)-\sin n(x-b)) \end{aligned}$$

The Fourier series of f becomes

$$\frac{b-a}{2\pi}+\sum_{n\geq 1}\frac{1}{\pi n}(\sin n(x-a)-\sin n(x-b)).$$

By the Dirichlet's Test (Ex 7b (p.60)), $\sum_{n\geq 1}\frac{\sin n(x-a)}{n}$ and $\sum_{n\geq 1}\frac{\sin n(x-b)}{n}$ converge for all x. Hence the Fourier series converges at every point x.

If $a=-\pi$ and $b=\pi$, then $\widehat{f}(n)=0$ for $n\neq 0$, then the Fourier series of f is $\frac{b-a}{2\pi}\equiv 1$ is equal to f itself.

Let's make some remarks. Firstly, note that we deal with a symmetric sum rather than an asymmetric one. Consider for example $a=-\pi$, $b=0$, $x=\pi$. Then $\sum\frac{e^{in(x-b)}}{n}=\sum\frac{(-1)^n}{n}$ converges, but $\sum\frac{e^{in(x-a)}}{n}=\sum\frac{1}{n}$ diverges if it is an asymmetric sum. Secondly[§], to show that $\left|\sum_1^N\sin n\theta\right|$, $\left|\sum_1^N\cos n\theta\right|$ is bounded for "good" θ, observe that they are the absolute values of the real and imaginary part of

$$\sum_1^N e^{in\theta},$$

so they are less than or equal to

$$\left|\sum_1^N e^{in\theta}\right|=\left|\frac{e^{i\theta}-e^{i(N+1)\theta}}{1-e^{i\theta}}\right|\leq\frac{\left|e^{i\theta}\right|+\left|e^{i(N+1)\theta}\right|}{\left|1-e^{i\theta}\right|}=\frac{2}{\left|1-e^{i\theta}\right|}.$$

Finally, please note textbook Ch3 Theorem 2.1.

Ex15. (1 marks) Let $w=e^{ix}$. We assume $w\neq 1$. Then $D_k(x)=\sum_{\ell=-k}^{k}w^\ell=\frac{w^{-k}(1-w^{2k+1})}{1-w}=\frac{w^{-k}}{1-w}-\frac{w^{k+1}}{1-w}$. Hence

$$\begin{aligned} NF_N(x) &= \sum_{k=-(N-1)}^{0}\frac{w^k}{1-w}-\sum_{k=0}^{N-1}\frac{w^{k+1}}{1-w}=\frac{w^{-(N-1)}(1-w^N)}{(1-w)^2}-\frac{w(1-w^N)}{(1-w)^2} \\ &= \frac{w(w^{-N}-2+w^N)}{(1-w)^2}=\frac{(w^{N/2}-w^{-N/2})^2}{[w^{-1/2}(1-w)]^2}=\frac{(w^{N/2}-w^{-N/2})^2}{(w^{-1/2}-w^{1/2})^2}=\frac{\sin^2(Nx/2)}{\sin^2(x/2)}. \end{aligned}$$

The result follows.

[§] A student provides this idea.

Ex17. (a). (3 marks) For each fixed r, $P_r(\cdot)$ is an even function. Therefore

$$\frac{1}{2\pi}\int_{-\pi}^{0} P_r(\xi)d\xi = \frac{1}{2\pi}\int_{0}^{\pi} P_r(\xi)d\xi.$$

Since their sum is 1, they equal 1/2. We then decompose $A_r f(\theta)$ as

$$A_r f(\theta) = \frac{1}{2\pi}\int_{-\pi}^{0} f(\theta-\xi)P_r(\xi)d\xi + \frac{1}{2\pi}\int_{0}^{\pi} f(\theta-\xi)P_r(\xi)d\xi.$$

For any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $0 < \xi < \delta$, we have $|f(\theta^-) - f(\theta-\xi)| < \varepsilon$. For this δ, there exists $r_0 < 1$ such that for all $r_0 \le r < 1$, we have

$$\int_{\delta}^{\pi} P_r(\xi)d\xi < \varepsilon.$$

With $P_r(\cdot)$ non-negative and f bounded on $\mathbb{T}$ (since f is Riemann integrable), we see that

$$\begin{aligned}
\left|\frac{1}{2\pi}\int_{0}^{\pi} f(\theta-\xi)P_r(\xi)d\xi - \frac{f(\theta^-)}{2}\right| &= \left|\frac{1}{2\pi}\int_{0}^{\pi}\left[f(\theta-\xi) - f(\theta^-)\right]P_r(\xi)d\xi\right| \\
&\le \frac{1}{2\pi}\int_{0}^{\delta}\left|f(\theta-\xi) - f(\theta^-)\right|P_r(\xi)d\xi \\
&\qquad + \frac{1}{2\pi}\int_{\delta}^{\pi}\left|f(\theta-\xi) - f(\theta^-)\right|P_r(\xi)d\xi \\
&\le \frac{1}{2\pi}\int_{0}^{\delta}\varepsilon P_r(\xi)d\xi + \frac{2\sup f}{2\pi}\int_{\delta}^{\pi} P_r(\xi)d\xi \\
&< \varepsilon + \frac{\sup f}{\pi}\varepsilon.
\end{aligned}$$

This shows

$$\lim_{r\uparrow 1}\frac{1}{2\pi}\int_{0}^{\pi} f(\theta-\xi)P_r(\xi)d\xi = \frac{f(\theta^-)}{2}.$$

Similarly,

$$\lim_{r\uparrow 1}\frac{1}{2\pi}\int_{-\pi}^{0} f(\theta-\xi)P_r(\xi)d\xi = \frac{f(\theta^+)}{2}.$$

The result follows.

Alternative[¶]

[¶] A student provides this solution.

Since $f(\theta - x) + f(\theta + x)$ and $\frac{1-r^2}{1+r^2-2r\cos x}$ are even functions,

$$\begin{aligned} A_r(f)(\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos x} f(\theta - x)\mathrm{d}x \\ &= \frac{1}{2\pi}\left(\int_{0}^{\pi} \frac{1-r^2}{1+r^2-2r\cos x} f(\theta + x)\mathrm{d}x + \int_{0}^{\pi} \frac{1-r^2}{1+r^2-2r\cos x} f(\theta - x)\mathrm{d}x\right) \\ &= \frac{1}{2\pi}\int_{0}^{\pi} \frac{1-r^2}{1+r^2-2r\cos x} \left(f(\theta - x) + f(\theta + x)\right)\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos x} \frac{f(\theta - x) + f(\theta + x)}{2}\mathrm{d}x \text{ for } 0 \le r < 1. \end{aligned}$$

Let $g(t) = \begin{cases} \frac{f(\theta^+)+f(\theta^-)}{2} & \text{if } t = \theta \\ \frac{f(t)+f(2\theta - t)}{2} & \text{otherwise} \end{cases}$. Then $\lim\limits_{\substack{h\to 0\\ h>0}} g(\theta + h) = \lim\limits_{\substack{h\to 0\\ h>0}} \frac{f(\theta+h)+f(\theta-h)}{2} = \frac{f(\theta^+)+f(\theta^-)}{2} = g(\theta)$. Similarly $\lim\limits_{\substack{h\to 0\\ h>0}} g(\theta - h) = g(\theta)$. So g is continuous at θ. Therefore $\lim\limits_{r\to 1} A_r(f)(\theta) = \lim\limits_{r\to 1} A_r(g)(\theta) = g(\theta) = \frac{f(\theta^+)+f(\theta^-)}{2}$.

(b) Since the Fejér kernel is also a good kernel and it is an even function, we can prove the result by applying the same procedure as in (a). We therefore omit the detail.