TA's solution* to 3093 assignment 1

1. (4 marks) f is periodic with period 2π because $f(x+2\pi) = e^{inx}e^{i2\pi} = e^{inx} = f(x)$.

If $n=0$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx = 1$. If $n \neq 0$, then $e^{-in\pi} = e^{-in\pi+2in\pi} = e^{in\pi}$ and hence we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}dx = \frac{1}{2\pi in}[e^{inx}]_{-\pi}^{\pi} = \frac{e^{in\pi}-e^{-in\pi}}{2\pi in} = 0.$$

For the second part, we note that $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, so

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cdot \cos mx dx = \frac{1}{4\pi}\int_{-\pi}^{\pi} e^{i(n+m)x} + e^{-i(n+m)x} + e^{i(n-m)x} + e^{-i(n-m)x} dx$$

As $n, m \geq 1$, $n+m \neq 0$. By the above results,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cdot \cos mx dx = \begin{cases} 0, & \text{if } n \neq m; \\ 1, & \text{if } n = m. \end{cases}$$

Similarly,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \cdot \sin mx dx = \frac{-1}{4\pi}\int_{-\pi}^{\pi} e^{i(n+m)x} + e^{-i(n+m)x} - e^{i(n-m)x} - e^{-i(n-m)x} dx$$
$$= \begin{cases} 0, & \text{if } n \neq m; \\ 1, & \text{if } n = m. \end{cases}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \cdot \cos mx dx = \frac{1}{4\pi i}\int_{-\pi}^{\pi} e^{i(n+m)x} - e^{-i(n+m)x} + e^{i(n-m)x} - e^{-i(n-m)x} dx$$
$$= 0.$$

2(a). (3 marks)† In polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$. We have $U(r,\theta) = u(r\cos\theta, r\sin\theta)$, whence

$$\begin{aligned}
U_r &= u_x x_r + u_y y_r = u_x \cos\theta + u_y \sin\theta, \\
U_{rr} &= \frac{\partial u_x}{\partial r}\cos\theta + \frac{\partial u_y}{\partial r}\sin\theta \\
&= (u_{xx}x_r + u_{xy}y_r)\cos\theta + (u_{yx}x_r + u_{yy}y_r)\sin\theta \\
&= u_{xx}\cos^2\theta + 2u_{xy}\sin\theta\cos\theta + u_{yy}\sin^2\theta,
\end{aligned}$$

$$\begin{aligned}
U_\theta &= u_x x_\theta + u_y y_\theta = -u_x r\sin\theta + u_y r\cos\theta, \\
U_{\theta\theta} &= -(u_{xx}x_\theta + u_{xy}y_\theta)r\sin\theta - u_x r\cos\theta + (u_{yx}x_\theta + u_{yy}y_\theta)r\cos\theta - u_y r\sin\theta \\
&= u_{xx}r^2\sin^2\theta - 2u_{xy}r^2\sin\theta\cos\theta - u_x r\cos\theta + u_{yy}r^2\cos^2\theta - u_y r\sin\theta.
\end{aligned}$$

Hence, combining all the above formulas, we have $U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = u_{xx} + u_{yy} = 0$.

*This solution is adapted from the work by former TAs.

†Please refer to textbook Ch.1 Sec.2 p.20-22 for more information about this question.

2(b). (3 marks)[‡] Let $U(r,\theta) = F(r)G(\theta)$ and put it into the polar Laplace equation. We have

$$(F'' + \frac{1}{r}F')G + \frac{1}{r^2}FG'' = 0,$$

whence

$$\frac{r^2F''(r) + rF'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)}. \tag{1}$$

The left hand side of (1) is a function of r while the right hand side of (1) is a function of θ, so the expression must be a constant α. Hence, we obtain $G''(\theta) + \alpha G(\theta) = 0$. From elementary ODE, We have three cases

$$\begin{cases} G(\theta) = A_\alpha \cos(\sqrt{\alpha}\theta) + B_\alpha \sin(\sqrt{\alpha}\theta) & \text{(if } \alpha > 0) \\ G(\theta) = A_\alpha \theta + B_\alpha & \text{(if } \alpha = 0) \\ G(\theta) = A_\alpha \cosh(\sqrt{-\alpha}\theta) + B_\alpha \sinh(\sqrt{-\alpha}\theta) & \text{(if } \alpha < 0) \end{cases}$$

Note that G must be 2π periodic, so $\alpha = n^2$, where $n = 0, 1, 2, \ldots$.

(i) If $n = 0$, then $G(\theta) = B_0$. We have from (1)

$$r^2F''(r) + rF'(r) = 0.$$

Solving it using separation of variables, $F(r) = C\ln r + D$. However, $\ln r$ is unbounded when r goes to zero but the solution is bounded in the origin, hence we have $C = 0$ and $F(r) = D$. The solution is of the desired form.

(ii) If $n > 0$, then $\alpha = n^2$ and $G(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$. Putting back to (1), we have

$$r^2F''(r) + rF'(r) - n^2F(r) = 0.$$

This ODE is of 2nd order and linear, so the solution space is of dimension 2. We can check that r^n and r^{-n} are linearly independent solutions to it, so F is given by

$$F(r) = C_nr^n + D_nr^{-n}.$$

Since the solution is bounded in the origin, we have $D_n = 0$ and $F(r) = C_nr^n$. Hence, the solution is given by

$$U(r,\theta) = F(r)G(\theta) = r^n(A_n\cos(n\theta) + B_n\sin(n\theta)).$$

[‡]This part is largely based on the work by former TAs as I am not knowledgeable about differential equations to be honest.