Solution 6

1. Let $C^k[a,b]$ be the normed space consisting of all k-many times continuously differentiable functions under the norm
$$\|f\|_k = \sum_{j=0}^{k} \|f^{(j)}\|_\infty \ .$$
Show that
$$\rho(f,g) = \sum_{j=0}^{\infty} \frac{1}{2^j} \frac{\|f-g\|_j}{1+\|f-g\|_j} \ ,$$
defines a metric on $C^\infty[a,b]$, the space of smooth functions.

Solution. Done in class.

2. Let H be the collection of all closed, bounded nonempty sets in a metric space (X,d). For $A, B \in H$, define
$$\rho(A,B) = \sup\{d(a,B) : \ a \in A\},$$
where
$$d(a,B) = \inf\{d(a,b) : \ b \in B\}.$$

   (a) Show that $\rho(A,B) = 0$ if and only if $A \subset B$.

   (b) Show that $\rho(A,B) \le \rho(A,C) + \rho(C,B)$, $\forall A, B, C \in H$.

   (c) Verify that $d_H(A,B) \equiv \max\{\rho(A,B), \rho(B,A)\}$ defines a metric on H. (It is called the Hausdorff metric.)

Solution. I only provide the proof of (b). For $a \in A, d(a,b) \le d(a,c) + d(c,b)$, $\forall b, c$. Given $\varepsilon > 0$, there is some $c' \in C$ such that $d(a,c') \le d(a,C) + \varepsilon$. So $d(a,B) \le d(a,b) \le d(a,C) + \varepsilon + d(c',B) \le d(a,C) + \varepsilon + \rho(C,B) \le \rho(A,C) + \varepsilon + \rho(C,B)$. Taking sup over $a \in A$, we finally get $\rho(A,B) \le \rho(A,C) + \varepsilon + \rho(C,B)$. Now the triangle inequality comes from letting $\varepsilon \to 0$.

3. Determine whether $\mathbb{Z}$ and $\mathbb{Q}$ are complete sets in $\mathbb{R}$.

Solution. $\mathbb{Z}$ is a closed subset so it is complete. On the other hand, the closure of $\mathbb{Q}$ is $\mathbb{R}$, it is not complete.

4. Does the collection of all differentiable functions on $[a,b]$ form a complete set in $C[a,b]$?

Solution. No. Since $C[a,b]$ is complete, it suffices to show that the set of differentiable functions is not closed. But this is easy, I leave you to verify the sequence of differentiable functions $f_n(x) = (1/n + x^2)^{1/2}$ in $C[-1,1]$ converges uniformly to the non-differentiable function $f(x) = |x|$.

5. Let (X,d) be a metric space and $C_b(X)$ the vector space of all bounded, continuous functions in X. Show that it forms a complete metric space under the sup-norm. This problem will be used in the next problem.

Solution. Let $\{f_n\}$ be a Cauchy sequence in $C_b(X)$. For $\varepsilon > 0$, there exists n_1 such that
$$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty < \varepsilon, \quad \forall x \in X. \tag{1}$$

It shows that $\{f_n(x)\}$ is a numerical Cauchy sequence, so $\lim_{n\to\infty} f_n(x)$ exists. We define $f(x) = \lim_{n\to\infty} f_n(x)$. We check it is continuous at x_0 as follows. By passing $m \to \infty$ in (1), we have

$$|f(x)-f(x_0)| \le |f(x)-f_{n_1}(x)|+|f_{n_1}(x)-f_{n_1}(x_0)|+|f_{n_1}(x_0)-f(x_0)| \le 2\varepsilon+|f_{n_1}(x)-f_{n_1}(x_0)|.$$

As f_{n_1} is continuous, there is some δ such that $|f_{n_1}(x) - f_{n_1}(x_0)| < \varepsilon$ for $x \in B_\delta(x_0)$. It follows that we $|f(x) - f(x_0)| < 3\varepsilon$ for $x \in B_\delta(x_0)$, so f is continuous at x_0. Now, letting $m \to \infty$ in (1), we get $|f_n(x) - f(x)| \le \varepsilon$ for all $n \ge n_1$, so $f_n \to f$ uniformly. In particular, it means f is bounded.

6. We define a metric on $\mathbb{N}$, the set of all natural numbers by setting

$$d(n,m) = \left|\frac{1}{n} - \frac{1}{m}\right| .$$

   (a) Show that it is not a complete metric.

   (b) Describe how to make it complete by adding one new point.

   Solution. The sequence $\{n\}$ is a Cauchy sequence in this metric but it has no limit. Its completion is obtained by adding an ideal point called ∞ and define $\tilde{d}(x,y) = d(x,y)$ when $x, y \in \mathbb{N}$ and $\tilde{d}(x,\infty) = 1/x$ for all $x \in \mathbb{N}$ and $\tilde{d}(\infty,\infty) = 0$.

7. Let (X,d) be a metric space. Fixing a point $p \in X$, for each x define a function

$$f_x(z) = d(z,x) - d(z,p).$$

   (a) Show that each f_x is a bounded, uniformly continuous function in X.

   (b) Show that the map $x \mapsto f_x$ is an isometric embedding of (X,d) to $C_b(X)$ (shorthand for $C_b(X,\mathbb{R})$) . In other words,

$$\|f_x - f_y\|_\infty = d(x,y), \quad \forall x, y \in X.$$

   (c) Deduce from (b) the completion theorem.

   This approach is much shorter than the proof given in notes. However, it is not so inspiring.

   Solution.

   (a) From $|f_x(z)| = |d(z,x) - d(z,p)| \le d(x,p)$, and from $|f_x(z) - f_x(z')| \le |d(z,x) - d(z',x)| + |d(z',p) - d(z,p)| \le 2d(z,z')$, it follows that each f_x is a bounded, uniformly continuous function in X.

   (b) $|f_x(z) - f_y(z)| = |d(z,x) - d(z,y)| \le d(x,y)$, and equality holds taking $z = x$. Hence

$$\|f_x - f_y\|_\infty = d(x,y), \quad \forall x, y \in X.$$

   (c) Let $Y_0 = \{f_x : x \in X\} \subset C_b(X)$. Let Y be the closure of Y_0 in the complete metric space $(C_b(X), \rho)$ with sup-norm ρ. Then (Y, ρ) is a completion of (X,d).

8. Let $f : E \to Y$ be a uniformly continuous map where $E \subset X$ and X, Y are metric spaces. Suppose that Y is complete. Show that there exists a uniformly continuous map F from $\overline{E}$ to Y satisfying $F = f$ in E. In other words, f can be extended to the closure of E preserving uniform continuity.

   Solution. Let $x \in \partial E$. There exists $\{x_n\} \subset E, x_n \to x$. Since $\{x_n\}$ is a Cauchy sequence, by uniformly continuity $\{f(x_n)\}$ is also a Cauchy sequence in Y. As Y is complete, $\{f(x_n)\}$

converges to some point in Y. Therefore, we can define $F(x) = \lim_{n\to\infty} f(x_n)$. It remains to show this definition is independent of the sequence $\{x_n\}$. Indeed, let $\{y_n\}, y_n \to x$. We claim $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(x_n)$. It suffices to set $z_{2n+1} = x_n$ when n is odd and $z_{2n} = y_n$ to form a new sequence $\{z_n\}$. This sequence again is a Cauchy sequence, so $\{f(z_n)\}$ is convergent. As both $\{x_n\}$ and $\{y_n\}$ are subsequences of it, $\{f(x_n)\}$ and $\{f(y_n)\}$ tend to the same limit. Now, it is clear that the new function F extends f and is uniformly continuous on the closure of E.

Note. We have used this property in the proof of Theorem 3.4. Observe that a contraction is always uniformly continuous.