Solution 5

1. In class we showed that the set $P = \{f : f(x) > 0, \forall x \in [a,b]\}$ is an open set in $C[a,b]$. Show that it is no longer true if the norm is replaced by the L^1-norm. In other words, for each $f \in P$ and each $\varepsilon > 0$, there is some continuous g which is negative somewhere such that $\|g - f\|_1 < \varepsilon$.

   Solution. Fix a point, say, a and consider the continuous piecewise function φ_k which is equal to 1 at a and vanishes on $[a + 1/k, b]$. Then
$$\int_a^b \varphi_k(x)dx = \frac{1}{2k} \ .$$
   Let $f \in C[a,b]$ and $g_k = f - (f(a)+1)\varphi_k$ also belongs to $C[a,b]$ and $g_k(a) = -1 < 0$, but
$$\|f - g_k\|_1 = \int_a^b |f(x) - g_k(x)|dx = \frac{f(a)+1}{2k} \to 0$$
   as $k \to \infty$.

2. Show that $[a,b]$ can be expressed as the intersection of countable open intervals. It shows in particular that countable intersection of open sets may not be open.

   Solution. Simply observe
$$[a,b] = \bigcap_{j=1}^{\infty} (a - 1/j, b + 1/j) \ .$$

3. Optional. Show that every open set in $\mathbb{R}$ can be written as a countable union of disjoint open intervals. Suggestion: Introduce an equivalence relation $x \sim y$ if x and y belongs to the same open interval in the open set and observe that there are at most countable many such intervals.

   Solution.

   Let V be open in $\mathbb{R}$. Fix x $\in V$, there exists some open interval I, x $\in I$, $I \subseteq V$. Let I_α $= (a_\alpha, b_\alpha)$, $\alpha \in \mathcal{A}$, be all intervals with this property. Let
$$I_x = (a_x, b_x), a_x = \inf_\alpha a_\alpha, b_x = \sup_\alpha b_\alpha.$$
   satisfy x $\in I_x$, $I_x \subseteq V$ (the largest open interval in V containing x). It is obvious that $I_x \cap I_y \neq \phi \Rightarrow I_x = I_y$. Let $x \sim y$ if $I_x = I_y$. Then one can show that $\sim$ is an equivalence relation. By the discussion above, we have
$$V = \bigcup_{x \in V} I_x = \bigcup_{[x] \in V/\sim} \left(\bigcup_{y \sim x} I_x \right) = \bigcup_{[x] \in V/\sim} I_x,$$
   which is a disjoint union. Moreover $V/\sim$ is at most countable since we can pick a rational number in each I_x to represent the class $[x] \in V/\sim$. Thus V can be written as a countable union of disjoint open intervals.

4. Identify the boundary points, interior points, interior and closure of the following sets in $\mathbb{R}$:

(a) $[1,2) \cup (2,5) \cup \{10\}$.
(b) $[0,1] \cap \mathbb{Q}$.
(c) $\bigcup_{k=1}^{\infty} (1/(k+1), 1/k)$.
(d) $\{1,2,3,\cdots\}$.

Solution.

(a) Boundary points: $1,2,5,10$. Interior points: $(1,2),(2,5)$. Interior: $(1,2)\cup(2,5)$. Closure: $[1,5]\cup\{10\}$.
(b) Boundary points: All points in $[0,1]$. No interior point. Interior: the empty set ϕ. Closure: $[0,1]$
(c) Boundary points: $\{1/k : \ k \geq 1\}, 0$. Interior points: all points in this set. Interior: This set (because it is an open set). Closure: $[0,1]$.
(d) Boundary points $1,2,3,\cdots$. No interior points. Interior: ϕ. Closure: the set itself (it is a closed set).

5. Identify the boundary points, interior points, interior and closure of the following sets in $\mathbb{R}^2$:

(a) $R \equiv [0,1) \times [2,3) \cup \{0\} \times (3,5)$.
(b) $\{(x,y) : 1 < x^2 + y^2 \leq 9\}$.
(c) $\mathbb{R}^2 \setminus \{(1,0),(1/2,0),(1/3,0),(1/4,0),\cdots\}$.

Solution.

(a) Boundary points: the geometric boundary of the rectangle and the segment $\{0\} \times [3,5]$. Interior points: all points inside the rectangle. Interior $(0,1)\times(3,5)$. Closure $[0,1]\times[3,5]\cup\{0\}\times[3,5]$.
(b) Boundary points: all (x,y) satisfying $x^2+y^2 = 1$ or $x^2+y^2 = 9$. Interior points: all points satisfying $1 < x^2+y^2 < 9$. Interior $\{(x,y) : \ 1 < x^2+y^2 < 9\}$. Closure $\{(x,y) : \ 1 \leq x^2+y^2 \leq 9\}$.
(c) Boundary points: $(0,0),(1,0),(1/2),(1/3,0),\cdots$. Interior points: all points except boundary points. Interior: $\mathbb{R}^2 \setminus \{(0,0),(1,0),(1/2),(1/3,0),\cdots\}$. Closure: $\mathbb{R}^2$.

6. Describe the closure and interior of the following sets in $C[0,1]$:

(a) $\{f : \ f(x) > -1, \ \forall x \in [0,1]\}$.
(b) $\{f : \ f(0) = f(1)\}$.

Solution.

(a) Closure: $\{f \in C[0,1] : \ f(x) \geq -1, \ \forall x \in [0,1]\}$. Interior: The set itself. It is an open set.
(b) Closure: The set itself. It is a closed set. Interior: ϕ. Let f satisfy $f(0) = f(1)$. For every $\varepsilon > 0$, it is clear we can find some $g \in C[0,1]$ satisfying $\|g - f\|_\infty < \varepsilon$ but $g(0) \neq g(1)$. It shows that every metric ball $B_\varepsilon(f)$ must contain some functions lying outside this set.

7. Let A and B be subsets of (X, d). Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Does $\overline{A \cap B} = \overline{A} \cap \overline{B}$?

   Solution. We have $\overline{A} \subset \overline{B}$ whenever $A \subset B$ right from definition. So $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Conversely, if $x \in \overline{A \cup B}$, $B_\varepsilon(x)$ either has non-empty intersection with A or B. So there exists $\varepsilon_j \to 0$ such that $B_{\varepsilon_j}(x)$ has nonempty intersection with A or B, so $x \in \overline{A} \cup \overline{B}$.

   On the other hand, $\overline{A \cap B} = \overline{A} \cap \overline{B}$ is not always true. For instance, consider intervals (a, b) and (b, c). We have $\overline{(a, b)} \cap \overline{(b, c)} = \{b\}$ but $\overline{(a, b) \cap (b, c)} = \phi$. Or you take A to be the set of all rationals and B all irrationals. Then $\overline{A \cap B} = \overline{\phi} = \phi$ but $\overline{A} \cap \overline{B} = \mathbb{R}$!

8. Show that $\overline{E} = \{x \in X : \ d(x, E) = 0\}$ for every non-empty $E \subset X$.

   Solution. Let $A = \{x \in X : \ d(x, E) = 0\}$. Claim that A is closed. Let $x_n \to x$ where $x_n \in A$. Recalling that $x \mapsto d(x, E)$ is continuous, so $d(x, E) = \lim_{n \to \infty} d(x_n, E) = 0$, that is, $x \in A$. We conclude that A is a closed set. As it clearly contains E, so $\overline{E} \subset A$ since the closure of E is the smallest closed set containing E. On the other hand, if $x \in A$, then $B_{1/n}(x) \cap E \neq \phi$. Picking $x_n \in B_{1/n}(x) \cap E$, we have $\{x_n\} \subset E$, $x_n \to x$, so $x \in \overline{E}$.