Hand in no. 2, 4, 5 and 6 by November 7.

Assignment 8

1. Let f be continuously differentiable on $[a, b]$. Show that it has a differentiable inverse if and only if its derivative is not equal to 0 at every point. This is 2060 stuff.

2. Consider the function
$$f(x) = \frac{1}{2}x + x^2 \sin\frac{1}{x}, \quad x \neq 0,$$
and set $f(0) = 0$. Show that f is differentiable at 0 with $f'(0) = 1/2$ but it has no local inverse at 0. Does it contradict the Inverse Function Theorem?

3. Study the map on $\mathbb{R}^2$ given $(x, y) \mapsto (x^2 - y^2, 2xy)$. Show that it is local invertible everywhere except at the origin. Does its inverse exist globally?

4. Consider the mapping from $\mathbb{R}^2$ to itself given by $f(x, y) = x - x^2,\ g(x, y) = y + xy$. Show that it has a local inverse at $(0, 0)$. And then write down the inverse map so that its domain can be described explicitly.

5. Let F be a continuously differentiable map from the open $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$ whose Jacobian determinant is non-vanishing everywhere. Prove that it maps every open set in U to an open set, that is, F is an open map. Does its inverse $F^{-1} : F(U) \to U$ always exist?

6. Consider the function
$$h(x, y) = (x - y^2)(x - 3y^2), \quad (x, y) \in \mathbb{R}^2.$$
Show that the set $\{(x, y) :\ h(x, y) = 0\}$ cannot be expressed as a local graph of a C^1-function over the x or y-axis near the origin. Explain why the Implicit Function Theorem is not applicable.

7. Consider a real polynomial $p(x, \mathbf{a}) = a_0 + a_1x + \cdots + a_nx^n$ as a function of x and the coefficients. A point x_0 is a simple root of p if $p(x_0, \mathbf{a}) = 0$ and $p'(x_0, \mathbf{a}) \neq 0$ where $\mathbf{a} = (a_0, a_1, \cdots, a_n)$. Let x_0 be a simple of $p(\cdot, \mathbf{a}_0)$. Show that there is a smooth function φ defined in an open set in $\mathbb{R}^{n+1}$ containing $\mathbf{a}_0$ such that $x = \varphi(\mathbf{a})$ is a simple root for $p(\cdot, \mathbf{a}) = 0$. What happens when the root is simple.