Assignment 4

Hand in no. 5, 6, 10 and 11 by October 3, 2023.

1. Prove Hölder's Inequality in vector form: For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $p > 1$ and q conjugate to p,
$$|\mathbf{x} \cdot \mathbf{y}| \leq \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} \left(\sum_{j=1}^{n} |y_j|^q \right)^{1/q} .$$
You may prove it directly or deduce it from its integral form by choosing suitable functions f anf g.

2. Prove Minkowski's Inequality in vector form: For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $p > 1$,
$$\|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p .$$
You may prove it directly or deduce it from its integral form by choosing suitable functions f anf g.

3. Prove the generalized Hölder's Inequality: For $f_1, f_2, \cdots, f_n \in R[a, b]$,
$$\int_a^b |f_1 f_2 \cdots f_n| dx \leq \left(\int_a^b |f_1|^{p_1} \right)^{1/p_1} \left(\int_a^b |f_2|^{p_2} \right)^{1/p_2} \cdots \left(\int_a^b |f_n|^{p_n} \right)^{1/p_n} ,$$
where
$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1, \quad p_1, p_2, \cdots, p_n > 1 .$$

4. Show that for $1 \leq p < r \leq \infty$,

   (a)
$$\|\mathbf{x}\|_p \leq n^{\frac{1}{p} - \frac{1}{r}} \|\mathbf{x}\|_r ,$$

   (b)
$$\|\mathbf{x}\|_r \leq n^{\frac{1}{r}} \|\mathbf{x}\|_p .$$

5. Show that for $1 \leq p < r \leq \infty$, and$f \in C[a, b]$,
$$\|f\|_p \leq (b - a)^{\frac{1}{p} - \frac{1}{r}} \|f\|_r .$$

6. Show that there is no constant C such that $\|f\|_2 \leq C\|f\|_1$, for all $f \in C[0, 1]$.

7. Show that $\|\cdot\|_p$ is no longer a norm on $C[0, 1]$ for $p \in (0, 1)$.

8. In a metric space (X, d), its metric ball is the set $\{y \in X : \ d(y, x) < r\}$ where x is the center and r the radius of the ball. May denote it by $B_r(x)$. Draw the unit metric balls centered at the origin with respect to the metrics d_2, d_∞ and d_1 on $\mathbb{R}^2$.

9. Determine the metric ball of radius r in (X, d) where d is the discrete metric, that is, $d(x, y) = 1$ if $x \neq y$.

10. Consider the function Φ defined on $C[a, b]$

$$\Phi(f) = \int_a^b \sqrt{1 + f^2(x)}\, dx.$$

Show that it is continuous in $C[a, b]$ under both the supnorm and the L^1-norm.

11. Consider the function Ψ defined on $C[a, b]$ given by $\Psi(f) = f(x_0)$ where $x_0 \in [a, b]$ is fixed. Show that it is continuous in the supnorm but not in the L^1-norm. Suggestion: Produce a sequence $\{f_n\}$ with $\|f_n\|_1 \to 0$ but $f_n(x_0) = 1,\ \forall n$. Ψ is called an evaluation map.