# Assignment 11

1. Use Baire Category Theorem to show that transcendental numbers are dense in the set of real numbers.

2. A set $E$ in a metric space is called a perfect set if, for each point $x \in E$ and $r > 0$, the ball $B_r(x) \bigcap E$ contains a point different from $x$.

   (a) For each $x$ in the perfect set $E$, there exists a sequence in $E$ consisting of infinitely many distinct points converging to $x$.

   (b) Every complete perfect set is uncountable. Hint: Use Baire Category Theorem.

   (c) Is (b) true without completeness?

3. Optional. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$.

   (a) Show that $\|x\| \leq C\|x\|_2$ for some $C$ where $\|\cdot\|_2$ is the Euclidean metric.

   (b) Deduce from (a) that the function $x \mapsto \|x\|$ is continuous with respect to the Euclidean metric.

   (c) Show that the inequality $\|x\|_2 \leq C'\|x\|$ for some $C'$ also holds. Hint: Observe that $x \mapsto \|x\|$ is positive on the unit sphere $\{x \in \mathbb{R}^n : \ \|x\|_2 = 1\}$ which is compact.

   (d) Establish the theorem asserting any two norms in a finite dimensional vector space are equivalent.

4. Let $P$ be the vector space consisting of all polynomials. Show that we cannot find a norm on $P$ so that it becomes a Banach space.

5. Let $\mathcal{F}$ be a subset of $C(X)$ where $X$ is a complete metric space. Suppose that for each $x \in X$, there exists a constant $M$ depending on $x$ such that $|f(x)| \leq M, \ \forall f \in \mathcal{F}$. Prove that there exists an open set $G$ in $X$ and a constant $C$ such that $\sup_{x \in G} |f(x)| \leq C$ for all $f \in \mathcal{F}$. Suggestion: Consider the decomposition of $X$ into the sets $X_n = \{x \in X : |f(x)| \leq n, \ \ \forall f \in \mathcal{F}\}$.

6. Optional. A function is called non-monotonic if if is not monotonic on every subinterval. Show that all non-monotonic functions form a dense set in $C[a,b]$. Hint: Consider the sets

$$\mathcal{E}_n = \{f \in C[a,b] : \exists x \text{ such that } (f(y) - f(x))(y - x) \geq 0, \ \forall y, \ |y - x| \leq 1/n\}.$$