# MATH3030 Tutorial 9

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# 9 Product rings and the Chinese Remainder theorem

## 9.1 Definition and characterization of product rings

### 9.1.1 Product rings

Let $R, R'$ be rings. Then $R \times R' := \{(r, r') : r \in R, r' \in R'\}$ is a ring with component-wise addition and multiplication. The unity is $(1_R, 1_{R'})$.

We have two projections: $\pi_1 : R \times R' \to R$ by $\pi_1(r, r') = r$, and $\pi_2 : R \times R' \to R'$ by $\pi_2(r, r') = r'$. The two maps preserves identity, addition, and multiplication. The kernels are $0 \times R'$ and $R \times 0$ respectively.

In other word, we have two short exact sequences:

$$0 \longrightarrow 0 \times R' \longrightarrow R \times R' \xrightarrow{\pi_1} R \longrightarrow 0.$$

$$0 \longrightarrow R \times 0 \longrightarrow R \times R' \xrightarrow{\pi_2} R' \longrightarrow 0.$$

Note that $R \times 0$ is a ring with unity $e_1 = (1, 0)$, and it is isomorphic to $R$. But it is not a subring of $R \times R'$ because the unity of the two rings are not the same. Similar things hold for $0 \times R'$, which has unity $e_2 = (0, 1)$.

Note that $e_1^2 = e_1$. We say that an element with this property as $e_1$ is **idempotent**.

### 9.1.2 A characterization of product rings

In fact, in the commutative case, product rings are characterized by idempotent elements:

**Proposition 9.1.** *Let $S$ be a commutative ring. Let $e \in S$ be an idempotent element, that is, $e^2 = e$.*

1. *The element $e' = 1 - e$ is also idempotent, and $ee' = e'e = 0$.*

2. *$eS$ and $e'S$ are rings with identity elements $e$ and $e'$. Moreover, $m_e : S \to eS$ and $m_{e'} : S \to e'S$ are ring homomorphisms, where $m_a(s) = as$ for $a, s \in S$.*

3. *$S \simeq eS \times e'S$.*

PROOF.

## 9.2 The Chinese remainder theorem

**Theorem 9.2.** *Let $I, J \subseteq R$ be ideals, such that $I + J = R$. Then*

1. *$I \cap J = IJ$.*

2. *$R/IJ \simeq R/I \times R/J$.*

**Example.** 1. $\mathbb{Z}/(105) \simeq \mathbb{Z}/(3) \times \mathbb{Z}/(5) \times \mathbb{Z}/(7)$.
2. $\mathbb{Z}[i]/(5) \simeq \mathbb{F}_5[x]/(x^2+1) \simeq \mathbb{F}_5[x]/(x-2) \times \mathbb{F}_5[x]/(x+2) \simeq \mathbb{F}_5 \times \mathbb{F}_5$.
3. $\mathbb{Z}[i]/(13) \simeq \mathbb{F}_{13}[x]/(x^2+1) \simeq \mathbb{F}_{13}[x]/(x-5) \times \mathbb{F}_{13}[x]/(x+5) \simeq \mathbb{F}_{13} \times \mathbb{F}_{13}$.