THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2023-24
Tutorial 7
26th October 2023

- Tutorial exercise would be uploaded to blackboard on Tuesdays provided that there is a tutorial class on that Thursday. You are not required to hand in the solution, but you are advised to try the problems before tutorial classes.
- Please send an email to echlam@math.cuhk.edu.hk if you have any questions.

1. Let P be a normal Sylow p-subgroup of G, prove that for any $H \leq G$, $P \cap H$ is the unique Sylow p-subgroup of H.

2. Prove that a group of order $56 = 2^3 \cdot 7$ is not simple.

3. Suppose G is a simple group of order $168 = 2^3 \cdot 3 \cdot 7$, how many elements of order 7 does G contain?

4. Prove that a group of order $231 = 3 \cdot 7 \cdot 11$ has center $|Z(G)| \geq 11$. (Hint: Try to show that the Sylow 11-subgroup is contained in the center.)

5. Suppose G is an even order simple group, with $|G| = 2^r m$ for some odd m, assume that it has a cyclic Sylow 2-subgroup P. Denote $\phi : G \to S_{2^r m}$ to be the homomorphism associated to the left regular action of G on itself. Recall that symmetric group has a natural sign homomorphism $\mathrm{sgn} : S_{2^r m} \to \mathbb{Z}_2$ where it sends even permutations to 0 and odd permutations to 1.

   (a) Consider $\psi = \mathrm{sgn} \circ \phi : G \to \mathbb{Z}_2$, let $s \in P$ be a generator of a cyclic Sylow 2-subgroup, show that $\psi(s) = 1$.

   (b) Prove that ψ is in fact surjective and $G \cong \mathbb{Z}_2$.

6. Let G be a group that satisfies the following condition: for each $n \geq 1$,

$$|\{g \in G : \ g^n = 1\}| \leq n.$$

   (a) Prove that the Sylow p-subgroups of G are unique, for each prime p dividing $|G|$.

   (b) Prove that the Sylow p-subgroups of G are cyclic, for each prime p dividing $|G|$.

   (c) Conclude that G is cyclic.

7. Let G be a group of order pqr where $p < q < r$ are primes.

   (a) Prove that:
   - If $n_r \neq 1$, there are at least $pq(r-1)$ elements of order r.
   - If $n_q \neq 1$, there are at least $r(q-1)$ elements of order q.
   - If $n_p \neq 1$, there are at least $q(p-1)$ elements of order p.

   (b) Deduce that at least one of n_p, n_q, n_r must be one, explain why G must be solvable.

8. Let G be a group of order $2^n m$ where m is an odd integer, suppose that the Sylow 2-subgroup P is normal and cyclic, and G/P is again cyclic.

   (a) Prove that G is abelian.

   (b) Prove that G is cyclic.