THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2023-24
Tutorial 3
28th Semptember 2023

- Tutorial exercise would be uploaded to blackboard on Tuesdays provided that there is a tutorial class on that Thursdays. You are not required to hand in the solution, but you are advised to try the problems before tutorial classes.
- Please send an email to echlam@math.cuhk.edu.hk if you have any questions.

1. Let $\varphi : G \to G'$ be a surjective homomorphism, prove that if G is cyclic, then G' is also cyclic. And if G is abelian, then G' is abelian.
2. Let H be a subgroup of G, suppose $N \lhd G$, prove that $H \cap N \lhd H$.
3. Find an example of a group G and a subgroup H so that there is an element $a \in G$ with $aHa^{-1} \leq H$ but $aHa^{-1} \neq H$.
4. Let N, M be normal subgroups of G, suppose $N \cap M = \{e\}$, prove that for all $m \in M$ and $n \in N$, $mn = nm$.
5. Let $Z_1 \leq G_1$ and $Z_2 \leq G_2$ be the centers of G_1 and G_2 respectively, prove that $Z_1 \times Z_2$ is the center of $G_1 \times G_2$.
6. In the following exercise, we will try to find all normal subgroups of S_4.
   (a) Prove that for any $\sigma \in S_4$, $\sigma x \sigma^{-1}$ preserves the cycle type of x. For example, if x is a 4-cycle like (1324), then $\sigma x \sigma^{-1}$ is again a 4-cycle. (Hint: prove the statement for σ being transpositions (ij).)
   (b) The possible cycle types of S_4 corresponds to partition of the integer 4, i.e. the different ways of writing 4 as a sum of positive integers. For example $2 + 1 + 1 = 4$ corresponds to a 2-cycle like $(13)(2)(4) = (13)$. For S_4, there are 4-cycles, 3-cycles, 2-cycles, $(2, 2)$-cycles like $(13)(24)$ and the only 1-cycle identity. Find out the numbers of elements in each different cycle types. (E.g. there is only one element that is a 1-cycle.)
   (c) By (a), we know that any normal subgroup must be a union of all elements in certain cycle types. Using Lagrange's theorem, find out potential candidates of normal subgroups of S_4 and determine which one are actual subgroups.
7. Suppose N is a normal subgroup of G so that $\mathrm{ord}\,(N)$ and $[G : N]$ are relatively prime, prove that any element in $x \in G$ satisfying $x^{\mathrm{ord}\,(N)} = e$ must be in N.
8. Suppose G contains a normal cyclic subgroup C, prove that any subgroup of C must also be normal in G.
9. Let $G = GL(2, \mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \, a, b, c, d \in \mathbb{R}, \det A \neq 0 \right\}$ be the group of 2×2 invertible matrices, prove that the commutator subgroup G' is contained in $SL(2, \mathbb{R}) = \{A \in G : \, \det A = 1\}$.

10. Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R}, a \neq 0, c \neq 0 \right\}$ with group operation given by matrix multiplications. Prove that $H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$ forms a normal subgroup of G. Is G/H abelian? Can you determine the group structure of G/H?

11. Prove that the abelianization of free group on n generators F_n is isomorphic to $\mathbb{Z}^{\oplus n}$.