**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH 3030 Abstract Algebra 2023-24**
**Tutorial 2**
**21st September 2023**

- The tutorial solutions are written for reference and proofs will be sketched briefly. You should try to fill in the details as an exercise. Please send an email to echlam@math.cuhk.edu.hk if you have any further questions.

1. Recall that any homomorphism from $F_n$ to an arbitrary group $G$ is uniquely determined by the image of the $n$ generators. If $G$ is a finite group of order $|G| > 1$, then $\mathrm{Hom}(F_n, G) \cong \mathrm{Hom}(\{1, 2, ..., n\}, G) \cong G^n$, which has cardinality $|G|^n$. In particular if $m \neq n$, the cardinalities of the sets of homomorphisms from $F_n$ to $G$ are not the same, so they are not isomorphic.

2. By Nielsen-Schreier theorem, $\langle x, y\rangle$ is a subgroup that is free. This is also abelian by assumption on $x, y$, therefore $\langle x, y\rangle \cong \mathbb{Z}$, i.e. there exists some $c$ such that $\langle c\rangle = \langle x, y\rangle$, so that $x = c^i$ and $y = c^j$.

   (Let me stress once again that this is outside of the syllabus and only serves as a demonstration of the powerful theorem.)

3. (a) Suppose $F_2 = F(a, b)$, let $x_i = a^i b^i$ for any $1 \leq i \leq m$, then we can show that $\langle x_1, ..., x_m\rangle \cong F_m$ for any $m \in \mathbb{Z}_{>0}$. Write $F_m = F(\{a_1, ..., a_m\})$, then as discussed before there is a homomorphism $\varphi : F_m \to \langle x_1, ..., x_m\rangle$ by sending $a_i \mapsto x_i$. This is by definition surjective, since $\langle x_1, ..., x_m\rangle$ contains words in $x_i$'s which is the image of the corresponding words in $a_i$'s. So it suffices to prove that $\varphi$ is injective.

   Injectivity amounts to showning that there are no relations among the $x_i$'s. In other words, if $w \in F_m$ is a word in $a_i$'s so that $\varphi(w)$ reduces to the empty word in $F(a, b)$, then the unreduced word $\varphi(w)$, expressed in terms of $a, b$ would contain subwords like $b^k b^{-k}$. This is because the positive powers of $x_i$ has $b$'s at the end and the negative powers of $x_i$ has $b$'s in front. So in order for them to cancel out, we must have $x_k$ and $x_k^{-1}$ next to each other in $\varphi(w)$. So by an induction argument of the length of $w$, the word $w$ itself must originally reduce to the empty word. This completes the proof.

   Note that this shows that there is in fact $F(\mathbb{Z}_{>0}) \leq F_2$ where $F(\mathbb{Z}_{>0})$ is a free group of countably many generators.

   (b) Denote $[G]$ the underlying set of a group, then $F([G])$ is a free group whose generators are given by elements of $G$ without relations, we have a natural homomorphism $\varphi : F([G]) \to G$ by sending $g \to g$. This is clearly surjective and by first isomorphism theorem $G \cong F([G])/\ker\varphi$.

   (c) We can look for normal subgroups by considering kernels of homomorphisms. For example, consider $F(a, b) \to \mathbb{Z} \times \mathbb{Z}$ by $a \mapsto (1, 0)$ and $b \mapsto (0, 1)$. Then the kernel of this homomorphism is given by the set of elements $w$ so that we have same numbers of $a$ and $a^{-1}$, $b$ and $b^{-1}$ appearing in the word $w$.

Inspired by the above, we see that it is not difficult to describe subgroups by specifying the degrees on the words. For example,

$$H = \{w = a^{i_1}b^{j_1}\cdots a^{i_n}b^{j_n} : i_1 + ... + i_n \text{ is divisible by } 2\}$$

gives another example of a normal subgroup, because conjugating a word $gwg^{-1}$ would not change the degree. Alternatively, you can also see that it is normal because it has index $[F(a,b) : H] = 2$. In this example, this corresponds to kernel of homomorphism $F(a,b) \to \mathbb{Z}_2$ by $a \mapsto 1$ and $b \mapsto 0$.

On the other hand, non-normal subgroups are easy to write down. For example, the subgroup given in part (a) for $m = 1$, i.e. the subgroup generated by $ab \in F(a,b)$. Clearly this group is isomorphic to $\mathbb{Z}$, and every element is given by $(ab)^k$ for some $k \in \mathbb{Z}$. So $a^{-1}aba = ba$ cannot be in the subgroup. Similarly $\langle a\rangle \leq F(a,b)$ is not normal.

4. Let $a \in G$, since $x \mapsto ax$ defines a set bijection from $G \to G$, we have $aNa^{-1} = \bigcap_{g\in G} agHg^{-1}a^{-1} = \bigcap_{ag=g'\in G} g'Hg'^{-1} = N$.

5. If $[G : H] < \infty$, there exists finitely many left coset space $aH$. Suppose $aH = bH$, then there is some $h$ so that $a = bh$, so $aHa^{-1} = bhHh^{-1}b^{-1} = bHb^{-1}$. Therefore there are at most as many subgroup of the form $gHg^{-1}$ as there are cosets $gH$, which is finite.

6. Contraposition of the implication is given by $aH = bH \Rightarrow Ha = Hb$. We can rewrite the implication equivalently as $b^{-1}aH = H \Rightarrow H = Hba^{-1}$. Now for any $g \in G$ and $h \in H$, choose $b = gh$ and $a = g$, then $b^{-1}aH = h^{-1}g^{-1}gH = h^{-1}H = H$ is fulfilled, so by assumption $H = Hba^{-1} = Hghg^{-1}$. By property of cosets we have $ghg^{-1} \in H$. Since $h \in H$ is arbitrary, we obtain $gHg^{-1} \leq H$ for any $g$.