THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2023-24
Tutorial 2
21st September 2023

- Tutorial exercise would be uploaded to course webpage on Tuesdays provided that there is a tutorial class on the coming Thursday. You are not required to hand in the solutions, but you are advised to try the problems before tutorial classes.

- Please send an email to echlam@math.cuhk.edu.hk if you have any questions.

Some notes on free groups

Let A be a set, the free group on A, denoted by $F(A)$ is defined as the smallest group generated by A, so that it is "as free as possible". That is to say, the relations we impose on words in A are the ones that are necessary in order to form a group. As such, one can find many different group homomorphism with domain $F(A)$. Namely, by the universal property of free group, in order to define $\varphi : F(A) \to G$ for some group G, one only has to specify what φ sends A to.

Theorem 1. (Free-forgetful adjunction) Let $F(A)$ be the free group on the set A and G be any group, then there is a natural bijection,

$$\mathrm{Hom}_{\mathbf{Set}}(A, |G|) \cong \mathrm{Hom}_{\mathbf{Grp}}(F(A), G),$$

where the left side is the set of functions from A to the underlying set $|G|$ of the group G, and the right side is the set of group homomorphism from $F(A)$ to G.

Remark. The notation $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is a common way to denote the set of morphism from A to B in the category $\mathscr{C}$. In the above, the categories in concerns are $\mathbf{Set}$ the category of sets and $\mathbf{Grp}$ the category of groups, and the respective morphism are simply functions between sets and homomorphisms between groups. The bijection above is a consequence of $A \mapsto F(A)$ and $G \mapsto |G|$ being adjoint functors.

In particular, there are many homomorphisms coming out of $F(A)$. As a result, there is a huge supply of normal subgroups of $F(A)$. It is then reasonable to ask about the structures of subgroups of $F(A)$. It turns out, subgroups of free groups are themselves free groups.

Theorem 2. (Nielsen-Schreier theorem) Let $F(A)$ be a free group on a finite set A, then every subgroup $H \leq F(A)$ is again free. Furthermore, if $[F(A) : H] = k$ and $|A| = n$, then H is freely generated by $1 + k(n-1)$ elements. If $[F(A) : H] = \infty$, and H contains a nontrivial normal subgroup of $F(A)$, then H is infinitely generated.

The proof of this theorem goes beyond the scope of the course. In fact, there is a beautiful proof of this theorem using the theory of covering spaces, which is a fundamental gadget in algebraic topology. The formula for the numbers of generators, in this setting, relates to some topological invariants of certain graphs.

Instead of giving you a proof of this result, it is more instructive to see this theorem in action. It is in fact possible to find a set of free generators for the subgroup.

Example 3. Let $A = \{a, b\}$ so that $F(A) = F_2 = \langle a, b\rangle$. Consider the homomorphism $\varphi : F_2 \to \mathbb{Z}/2\mathbb{Z}$ by $\varphi(a) = 1$ and $\varphi(b) = 0$, and let $H = \ker\varphi$. Here H consists of all words where the sums of the powers of a's appearing in the words are even. To find a set of free generators for H, we start by picking a set S consisting of representatives for each coset of H in F_2 such that S is closed under taking prefixes of words, i.e. if $w_1 w_2 ... w_n$ is a reduced word in S, then $w_1 w_2 ... w_k$'s are in S. For example, we may take $S = \{1, a\}$. For each $g \in F_2$, denote $\bar{g} \in S$ the unique element so that $gH = \bar{g}H$. We claim that the subset consisting of non-identity elements of $\{sx\overline{sx}^{-1} | s \in S, x \in A\}$ gives a free generating set of H. Computing this yields $\{b, a^2, aba^{-1}\}$. This agrees with the theorem since $1 + 2(2-1) = 3$.

Note that this generating set depends on the representatives we chose, for example, if we take $S = \{1, ab\}$, then the generating set will be $\{ab^{-1}a^{-1}, b, aba\}$.

Problems

1. Let F_n be the free group on n letters, prove that $F_n \ncong F_m$ for $n \neq m$.
2. Let F be a free group, prove that if $x, y \in F$ commutes, then they are powers of a common element. (Hint: Consider $\langle x, y \rangle \subset F$.)
3. (a) Let $n > 1$, can you find a subgroup of F_2 that is isomorphic to F_n?
   (b) Using the concept of presentations, justify why every group can be written as a quotient of a free group.
   (c) Can you write down two normal subgroups and two non-normal subgroups of F_2?
4. Let $H \leq G$ be a subgroup, define $N = \bigcap_{g \in G} gHg^{-1}$, show that N is a normal subgroup of G.
5. Let G be an infinite group and H be a finite index subgroup, i.e. $[G : H] < \infty$, show that there is finitely distinct subgroup of the form gHg^{-1}.
6. Let $H \subset G$, suppose that H satisfies the property that $Ha \neq Hb$ implies $aH \neq bH$, prove that $gHg^{-1} \leq H$ for any $g \in G$.