THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2023-24
Homework 7
Due Date: 9th November 2023

Compulsory Part

1. Let G be a finite group, and suppose that there exist representatives $g_1, \ldots, g_r$ of the r distinct conjugacy classes in G such that $g_i g_j = g_j g_i$ for all i, j. Show that G is abelian.

   Proof. Consider G acting on itself by conjugations. Fix a particular g_i, then for any $x \in G$, we know that it is belongs to some conjugacy class, i.e. $gxg^{-1} = g_j$ for some g. By assumption g_j commutes with g_i and so $gxg^{-1} \in Z_G(g_i)$ the stabilizer of g_i. In other words, $x \in g^{-1} Z_G(g_i) g$. This implies that $x \in \bigcup_{g \in G} g^{-1} Z_G(g_i) g$. So we necessarily have $G = \bigcup_{g \in G} g^{-1} Z_G(g_i) g$.

   Recall from tutorial 6 question 4b that this only happens when $Z_G(g_i) = G$ because for a proper subgroup $H \lneq G$ there are at most $[G : H]$ distinct subgroups of the form gHg^{-1} and so the union contains at most $[G : H](|H| - 1) + 1 < |G|$ many elements.

   Therefore g_i commutes with all of G for any i, so the conjugacy class is just a singleton $\{g_i\}$. Since this holds for all representative of each individual conjugacy class, this implies that G is abelian.

2. Let G be a finite group and let primes p and $q \neq p$ divide $|G|$. Prove that if G has precisely one proper Sylow p-subgroup, then it must be a normal subgroup, and hence G is not simple.

   Proof. By the second Sylow theorem, any Sylow p-subgroups are conjugate to each other. If P is the unique proper Sylow p-subgroup of G, then gPg^{-1} is again a Sylow p-subgroup, which must be itself. So for arbitrary $g \in G$ we have $gPg^{-1} = P$, hence it is a proper normal subgroup.

3. Let G be a finite group and let p be a prime dividing $|G|$. Let P be a Sylow p-subgroup of G.

   (a) Show that P is the only Sylow p-subgroup of $N_G(N_G(P))$.

   (b) Using part (a) and applying Sylow Theorems, show that $N_G(N_G(P)) = N_G(P)$.

   Proof.

   (a) Recall that $N_G(P)$ is the normalizer of P, i.e. consisting of all g so that $gPg^{-1} = P$. By definition P is normal in $N_G(P)$ and so it is the unique Sylow p-subgroup of $N_G(P)$. Now let $h \in N_G(N_G(P))$, then $hN_G(P)h^{-1} = N_G(P)$. Restricting this on P, by order consideration hPh^{-1} must be some Sylow p-subgroup inside $N_G(P)$ so $hPh^{-1} = P$ by uniqueness. This implies that P is also normal in $N_G(N_G(P))$, so it is also the unique Sylow p-subgroup.

   (b) Recall that $N_G(P) = \{g \in G : gPg^{-1} = P\}$, since any $h \in N_G(N_G(P))$ satisfies this condition by part (a), we have $N_G(N_G(P)) = N_G(P)$.

4. Show that there are no simple groups of order $p^r m$, where p is a prime, r is a positive integer, and $1 < m < p$.

   Proof. Suppose G is a group of order $p^r m$ with $1 < m < p$, consider Sylow p-subgroups of G, by third Sylow theorem, $n_p = p^i k \equiv 1 \mod p$ for $0 \leq i \leq r$ and $1 \mid k \mid m$. This implies that i must be zero and $k = 1$. So $n_p = 1$ and there is a unique Sylow p-subgroup of G, which is proper and normal. So G cannot be simple.

5. Let G be a group of order 6. Suppose G is not abelian.

   (a) Show that G has three subgroups of order 2.

   (b) Show that there is a homomorphism $\phi : G \to S_3$ with $|\ker(\phi)| \leq 2$. [*Hint:* Consider the action of G on the set of left cosets of a subgroup of order 2 in G (as in HW6, Optional Q.5).]

   (c) Show that $G \simeq S_3$.

   Proof.

   (a) If G is of order 6 and nonabelian, then by third Sylow theorem $n_3 \equiv 1 \mod 3$ and $n_3 | 6$ imply that $n_3 = 1$. While $n_2 \equiv 1 \mod 2$ and $n_2 | 6$ imply that n_2 can be 1 or 3. If n_2 was also 1, then the Sylow subgroups are isomorphic to $\mathbb{Z}_2$ and $\mathbb{Z}_3$, and $G \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, which is abelian. This gives a contradiction. So $n_2 = 3$ and there are three subgroups of order 2.

   (b) Suppose P is one of the Sylow 2-subgroups, then G acts on the left coset space G/P by left multiplication. As $|G/P| = 3$, this group action induces a homomorphism $\phi : G \to S_3$. Since the action is transitive, the image of ϕ consists of (123). So $|\phi(G)| \geq 3$, or $\ker(\phi) \leq 2$.

   (c) Consider the non-identity element x of P, we would like to prove that x acting on left P-cosets nontrivially, i.e $\phi(x) \neq \text{id} \in S_3$. Assume for the sake of contradiction that $\phi(x) = \text{id}$, then for an arbitrary $y \in G$, y lies in some coset yP and we have $x \cdot yP = yP$ since we have assumed that x acts trivially. This implies that $y^{-1}xy \in P$. So we have $y^{-1}Py = P$. But since y is arbitrary, this would imply that P is a normal subgroup, contradicting the fact that P is not the unique Sylow 2-subgroup.

   Now x acts nontrivially on $G/P = \{P, g_1P, g_2P\}$, and x fixes P because $x \in P$. So it swaps the two other cosets. In other words, $\phi(x)$ is a 2-cycle in S_3. Since $\phi(G)$ contains a 2-cycle and a 3-cycle, it is the whole group. And $\phi : G \to S_3$ is an isomorphism.

6. (a) Let G be a finite group, and $H, K < G$. Show that

$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|}.$$

   (Note that HK may not be a subgroup of G, so the above is just an equality between orders of sets.)

   (b) Suppose that G is a finite group of order 48.

i. Applying Sylow Theorems, show that the number n_2 of Sylow 2-subgroups in G is either 1 or 3.

ii. Suppose that $n_2 = 3$ and let H, K be two distinct Syloew 2-subgroups in G. Show that $|H \cap K| = 8$ by applying part (a). From this and considering the normalizer $N_G(H \cap K)$, deduce that $H \cap K$ is normal in G, thereby showing that G cannot be simple.

Proof.

(a) Suppose $H \times K$ acts on HK by $(h, k) \cdot g = hgk^{-1}$. One can check that

$$(h_1, k_1)(h_2, k_2) \cdot g = h_1 h_2 g k_2^{-1} k_1^{-1} = (h_1 h_2) g (k_1 k_2)^{-1} = (h_1 h_2, k_1 k_2) \cdot g$$

And $(e, e) \cdot g = g$. Furthermore given $g, g' \in HK$, we can write $g = hk$ and $g' = h'k'$ and hence $g' = (h'h^{-1}, k'^{-1}k) \cdot g$. So this defines a transitive action. Note that any stabilizer (h, k) of $e \in HK$ satisfies $(h, k) \cdot e = hk^{-1} = e$. In other words, $(H \times K)_e = \{(h, k) \in H \times K : h = k\} \cong H \cap K$. By orbit-stabilizer theorem,

$$|HK| = \frac{|H \times K|}{|(H \times K)_e|} = \frac{|H| \cdot |K|}{|H \cap K|}.$$

(b) i. For a group G of order $48 = 2^4 \cdot 3$, third Sylow theorem implies that $n_2 \equiv 1 \mod 2$ and $n_2 | 48$. Since n_2 cannot be even, this forces $n_2 = 1$ or 3.

ii. If $n_2 = 3$, for two distinct Sylow 2-subgroups H, K, we have that $H \cap K$ is also a 2-group, with order possibly given by $8, 4, 2$ or 1. By considering part (a), $48 \geq |HK| = 16^2/2^k$ where $k = 3, 2, 1$ or 0. As $256/4 = 64 > 48$, the only possible value of $|H \cap K|$ is 8,. Now because $H \cap K$ has index 2 in H and K, it is a normal subgroup of both H and K. By definition $N_G(H \cap K)$ is the largest subgroup so that $H \cap K$ is normal in, so our observation implies that $H, K \leq N_G(H \cap K)$. Then $HK \subset N_G(H \cap K)$. Since $|HK| = 32$, we immediately have $N_G(H \cap K) = G$. And so $H \cap K$ is in fact a proper normal subgroup in G. So that G is not simple.

Optional Part

1. Let G be a finite group of odd order. Suppose that $g \in G$ and g^{-1} lie in the same conjugacy class. Show that $g = e$.

   Proof. Note that if $hgh^{-1} = g^{-1}$, then $hg^{-1}h^{-1} = (hgh^{-1})^{-1} = g$. We can let $\langle h \rangle$ acts on the set $X = \{g, g^{-1}\}$, which has cardinality 1 or 2 depending on whether $g = e$ or not. But $\langle h \rangle$ has odd order and the action is transitive, this forces $|X| = \frac{\operatorname{ord}(h)}{d}$ where d is the order of stabilizer of g, which is odd again. This implies that $|X|$ must be an odd number, hence it is equal to 1 and $g = e$.

2. Show that every group of order 30 contains a subgroup of order 15.

   Proof. Let G be a group of order 30, then $n_3 \equiv 1 \mod 3$ so there can be 1 or 10 Sylow 3-subgroups. Likewise $n_5 \equiv 1 \mod 5$, so there can be 1 or 6 Sylow 5-subgroups. It is impossible to have both $n_3 = 10$ and $n_5 = 6$ because the Sylow p-subgroups are cyclic and have trivial intersection. So having 10 Sylow 3-subgroups would give $10 \cdot 2 = 20$ elements of order 3, and having 6 Sylow 5-subgroups would give $4 \cdot 6 = 24$ elements of order 5, clearly exceeding the total number of elements in G.

As a result, either the Sylow 3-subgroup or the Sylow 5-subgroup is unique, and hence is normal. Say we have a normal Sylow 3-subgroup P, then if Q is any Sylow 5-subgroup, PQ is a subgroup of order $|PQ| = 15$, since they are cyclic and have trivial intersection. Same argument for the case when the Sylow 5-subgroup is unique.

3. Prove that no group of order 160 is simple.

   Proof. Let G be a group of order $160 = 2^5 \cdot 5$, then $n_2 \equiv 1 \mod 2$ so there can be 1 or 5 Sylow 2-subgroups. If there are a unique Sylow 2-subgroup, then it is proper normal and G cannot be simple.

   Now suppose $n_2 = 5$. Recall that G acts on the set of Sylow 2-subgroups T by conjugation. Since there are 5 such subgroups, we get a permutation homomorphism $G \to S_5$. Now G has order 160 while S_5 has order 120. The kernel of such map must be nontrivial and we obtain a proper normal subgroup of G.

4. How many elements of order 7 are there in a simple group of order 168?

   Proof. Let G be a simple group of order 168. Then $n_7 \mid 168$ and $n_7 \equiv 1 \pmod 7$. Then $n_7 \mid 24$, and so $n_7 = 1$ or 8. Since G is simple, $n_7 \neq 1$. Then $n_7 = 8$. Let $P_1, ..., P_8$ be the 8 subgroups of order 7 in G. Then $|P_i \cap P_j| = 1$ for each $i \neq j$. Each element of order 7 lies in precisely one of $P_1, ..., P_8$, and each of $P_1, ..., P_8$ contains 6 elements of order 7. Then there are $48 = 8 \cdot 6$ elements of order 7 in G.

5. Let p, q be prime numbers. Show that a group of order p^2q is solvable.

   Proof. Let p, q be prime numbers. Let G be a group of order p^2q.

   When $p = q$, it follows from Sylow I that G contains a subnormal series $\{e\} = H_0 < H_1 < H_2 < H_3 = G$, where each $|H_i| = p^i$. Then each H_{i+1}/H_i is cyclic of order p. Then G is solvable.

   When $p > q$, $n_p | q$ and $n_p \equiv 1 \pmod p$. Then $n_p = 1$. Let P be the unique Sylow p-subgroup of G. Then $P \lhd G$. Since $|P| = p^2$ and $|G/P| = q$, both P and G/P are abelian. Then G is solvable.

   When $p < q$, $n_q | p^2$ and $n_q \equiv 1 \pmod q$. Then $n_q = 1$ or $n_q = p^2$. In the former case, there is a unique Sylow q-subgroup Q of G. Then $Q \lhd G$, $|Q| = q$ and $|G/Q| = p^2$. Again, $Q, G/Q$ are both abelian, so G is solvable.

   In the later case, $q \mid p^2 - 1$, so $q \mid p - 1$ or $q \mid p + 1$. But $p < q$, so it must be that $q \mid p + 1$ and that $q = p + 1$. Then $p = 2$, $q = 3$, and $n_3 = 4$. As in the last question, there are 8 elements of order 3. Any group of order 4 must consist of the remaining 4 elements in G. Then there exists a unique Sylow 2 subgroup P of order 4. Then P and G/P are both abelian, so G is solvable.

6. Let $p < q < r$ be prime numbers. Show that a group of order pqr is not simple.

   Proof. Let $p < q < r$ be prime numbers. Let G be a group of order pqr that is simple. Then $n_p \mid qr$, $n_q \mid pr$, $n_r \mid pq$; $n_p \equiv 1 \pmod p$, $n_q \equiv 1 \pmod q$, $n_r \equiv 1 \pmod r$, and $n_p, n_q, n_r \neq 1$.

Then $n_r = pq$. Then there are $pq(r-1) = pqr - pq$ many elements of order r. Note that $n_q = r$ or pr. This gives at least $(q-1)r$ many elements of order q. Now $q-1 \geq p, r > q$, so $(q-1)r + pq(r-1) > pq + pq(r-1) = pqr = |G|$. This exceeds the number of elements of G. Contradiction arises.

Therefore, a group of order pqr is not simple.