THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2023-24
Homework 9
Due Date: 30th November 2023

Compulsory Part

1. Let R be a commutative ring and I an ideal of R. Show that the set $\sqrt{I}$ of all $a \in R$, such that $a^n \in I$ for some $n \in \mathbb{Z}^+$, is an ideal of R, called the **radical** of I.

2. Show by examples that for proper ideals I of a commutative ring R,

   (a) $\sqrt{I}$ need not equal I.

   (b) $\sqrt{I}$ may equal I.

3. Prove that $\mathbb{Z}[x]$ is not a PID by showing that the ideal $\langle 2, x\rangle$ is not principal.

4. Let D be an integral domain. Show that, for $k = 1, \ldots, n$, the ideal $\langle x_1, \ldots, x_k\rangle$ is prime in $D[x_1, \ldots, x_n]$.

5. Let $\varphi : R \to S$ be a homomorphism of commutative rings, and let $I \subset S$ be an ideal. Prove that if I is a prime ideal in S, then $\varphi^{-1}(I)$ is a prime ideal in R. Show by giving an exmple that, however, $\varphi^{-1}(I)$ is not necessarily maximal when I is maximal.

6. Let R be a commutative ring, and let P be a prime ideal of R. Suppose that 0 is the only zero-divisor of R contained in P. Show that R is an integral domain.

7. Show that every prime ideal in a *finite* commutative ring R is a maximal ideal.

Optional Part

1. An element a of a ring R is **nilpotent** if $a^n = 0$ for some $n \in \mathbb{Z}^+$.

   Show that the collection N of all nilpotent elements in a commutative ring R is an ideal, called the **nilradical** of R.

2. Show that the nilradical N of a commutative ring R is contained in *every* prime ideal of R. (Actually N is the intersection of all prime ideals in R.)

3. What is the relationship between the radical $\sqrt{I}$ of an ideal I in a commutative ring R and the nilradical of the quotient ring R/I? Explain your answer carefully.

4. Let F be a subfield of a field E.

   (a) For $\alpha_1, \ldots, \alpha_n \in E$, define the *evaluation map*

   $$\phi_{\alpha_1,\cdots,\alpha_n} : F[x_1, \cdots, x_n] \to E$$

   by sending $f(x_1, \ldots, x_n)$ to $f(\alpha_1, \ldots, \alpha_n)$. Show that $\phi_{\alpha_1,\cdots,\alpha_n}$ is a ring homomorphism. We say that $(\alpha_1, \cdots, \alpha_n) \in F^n$ is a zero of $f = f(x_1, \cdots, x_n)$ if $f(\alpha_1, \ldots, \alpha_n) = 0$, or equivalently, if $\phi_{\alpha_1,\cdots,\alpha_n}(f) = 0$.

   (b) Given a subset $V \subset F^n$, show that the set of polynomials $f \in F[x_1, \cdots, x_n]$ such that every element in V is a zero of f forms an ideal of $F[x_1, \cdots, x_n]$.

5. Prove the *equivalence* of the following two statements:
   Fundamental Theorem of Algebra: Every nonconstant polynomial in $\mathbb{C}[x]$ has a zero in $\mathbb{C}$.
   Nullstellensatz for $\mathbb{C}[x]$: Let $f_1(x), \ldots, f_r(x) \in \mathbb{C}[x]$ and suppose that every $\alpha \in \mathbb{C}$ that is a zero of all r of these polynomials is also a zero of a polynomial $g(x)$ in $\mathbb{C}[x]$. Then some power of $g(x)$ is in the smallest ideal of $\mathbb{C}[x]$ that contains the r polynomials $f_1(x), \ldots, f_r(x)$.