THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2023-24
Homework 8
Due Date: 23th November 2023

Compulsory Part

1. Prove that if D is an integral domain, then $D[x]$ is an integral domain.

2. Let D be an integral domain and x an indeterminate.

   (a) Describe the units in $D[x]$.

   (b) Find the units in $\mathbb{Z}[x]$.

   (c) Find the units in $\mathbb{Z}_7[x]$.

3. Let R be a commutative ring with unity of prime characteristic p. Show that the map $\phi_p : R \to R$ given by $\phi_p(a) = a^p$ is a ring homomorphism (called the **Frobenius homomorphism**).

4. Show that for p a prime, the polynomial $x^p + a$ in $\mathbb{Z}_p[x]$ is reducible for any $a \in \mathbb{Z}_p$.

5. Let $\sigma_m : \mathbb{Z} \to \mathbb{Z}_m$ be the natural reminder homomorphism sending a to the remainder of a when divided by m, for $a \in \mathbb{Z}$.

   (a) Show that the induced map $\overline{\sigma}_m : \mathbb{Z}[x] \to \mathbb{Z}_m[x]$ given by

   $$\overline{\sigma}_m(a_0 + a_1x + \cdots + a_nx^n) = \sigma_m(a_0) + \sigma_m(a_1)x + \cdots + \sigma_m(a_n)x^n$$

   is a homomorphism from $\mathbb{Z}[x]$ onto $\mathbb{Z}_m[x]$.

   (b) Show that if $f(x) \in \mathbb{Z}[x]$ and $\overline{\sigma}_m(f(x))$ both have degree n and $\overline{\sigma}_m(f(x))$ does not factor in $\mathbb{Z}_m[x]$ into two polynomials of degree less than n, then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

   (c) Use part (b) to show that $x^3 + 17x + 36$ is irreducible in $\mathbb{Q}[x]$.

6. Let $\phi : R \to R'$ be a ring homomorphism and let N be an ideal of R.

   (a) Show that $\phi(N)$ is an ideal of im ϕ.

   (b) Given an example to show that $\phi(N)$ need not be an ideal of R'.

   (c) Let N' be an ideal of R'. Show that $\phi^{-1}(N')$ is an ideal of R.

Optional Part

1. Let F be a field. An element ϕ of F^F is a **polynomial function on** F, if there exists $f(x) \in F[x]$ such that $\phi(a) = f(a)$ for all $a \in F$.
   (a) Show that the set P_F of all polynomial functions on F forms a subring of F^F.
   (b) Give an example to show that the ring P_F is not necessarily isomorphic to $F[x]$.

2. Give an example to show that, when F is a finite field, P_F and $F[x]$ do not even have the same number of elements.

3. Let F be a field of characteristic zero and let D be the formal polynomial differentiation map, i.e.

$$D(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) := a_1 + 2 \cdot a_2 x + \cdots + n \cdot a_n x^{n-1}.$$

   (a) Show that $D : F[x] \to F[x]$ is a group homomorphism from $(F[x], +)$ into itself. Is D a ring homomorphism?
   (b) Find the kernel of D.
   (c) Find the image of $F[x]$ under D.

4. Let A and B be ideals of a ring R. The **product** AB **of** A **and** B is defined by

$$AB = \left\{ \sum_{i=1}^{n} a_i b_i : a_i \in A, b_i \in B, n \in \mathbb{Z}^+ \right\}.$$

   (a) Show that AB is an ideal in R.
   (b) Show that $AB \subseteq (A \cap B)$.

5. Let A and B be ideals of a *commutative* ring R. The **quotient** $A : B$ **of** A **by** B is defined by

$$A : B = \{r \in R : rb \in A \text{ for all } b \in B\}.$$

   Show that $A : B$ is an ideal of R.

6. Let R and R' be rings and let N and N' be ideals of R and R', respectively. Let ϕ be a homomorphism of R into R'. Show that ϕ induces a natural homomorphism $\phi_* : R/N \to R'/N'$ if $\phi(N) \subseteq N'$.