THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2023-24
Homework 7
Due Date: 9th November 2023

Compulsory Part

1. Let G be a finite group, and suppose that there exist representatives $g_1, \ldots, g_r$ of the r distinct conjugacy classes in G such that $g_i g_j = g_j g_i$ for all i, j. Show that G is abelian.

2. Let G be a finite group and let primes p and $q \neq p$ divide $|G|$. Prove that if G has precisely one proper Sylow p-subgroup, then it must be a normal subgroup, and hence G is not simple.

3. Let G be a finite group and let p be a prime dividing $|G|$. Let P be a Sylow p-subgroup of G.

   (a) Show that P is the only Sylow p-subgroup of $N_G(N_G(P))$.

   (b) Using part (a) and applying Sylow Theorems, show that $N_G(N_G(P)) = N_G(P)$.

4. Show that there are no simple groups of order $p^r m$, where p is a prime, r is a positive integer, and $1 < m < p$.

5. Let G be a group of order 6. Suppose G is not abelian.

   (a) Show that G has three subgroups of order 2.

   (b) Show that there is a homomorphism $\phi : G \to S_3$ with $|\ker(\phi)| \leq 2$. [*Hint:* Consider the action of G on the set of left cosets of a subgroup of order 2 in G (as in HW6, Optional Q.5).]

   (c) Show that $G \simeq S_3$.

6. (a) Let G be a finite group, and $H, K < G$. Show that

   $$|HK| = \frac{|H| \cdot |K|}{|H \cap K|}.$$

   (Note that HK may not be a subgroup of G, so the above is just an equality between orders of sets.)

   (b) Suppose that G is a finite group of order 48.

   i. Applying Sylow Theorems, show that the number n_2 of Sylow 2-subgroups in G is either 1 or 3.

   ii. Suppose that $n_2 = 3$ and let H, K be two distinct Syloew 2-subgroups in G. Show that $|H \cap K| = 8$ by applying part (a). From this and considering the normalizer $N_G(H \cap K)$, deduce that $H \cap K$ is normal in G, thereby showing that G cannot be simple.

Optional Part

1. Let G be a finite group of odd order. Suppose that $g \in G$ and g^{-1} lie in the same conjugacy class. Show that $g = e$.
2. Show that every group of order 30 contains a subgroup of order 15.
3. Prove that no group of order 160 is simple.
4. How many elements of order 7 are there in a simple group of order 168?
5. Let p, q be prime numbers. Show that a group of order p^2q is solvable.
6. Let $p < q < r$ be prime numbers. Show that a group of order pqr is not simple.