THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2023-24
Homework 6
Due Date: 26th October 2023

Compulsory Part

1. Let X be a G-set. Show that G acts faithfully on X if and only if no two distinct elements of G have the same action on each element of X.

2. Let X be a G-set and let $Y \subseteq X$. Show that $G_Y := \{g \in G : gy = y \text{ for all } y \in Y\}$ is a subgroup of G.

3. Let G be the additive group of real numbers. Let the action of $\theta \in G$ on the real plane $\mathbb{R}^2$ be given by rotating the plane counterclockwise about the origin through θ radians. Let P be a point other than the origin in the plane.

   (a) Show that $\mathbb{R}^2$ is a G-set.

   (b) Describe geometrically the orbit containing P.

   (c) Find the group G_P.

4. Let H be a subgroup of G, and let L_H be the set of all left cosets of H in G. Show that there is a well-defined action of G on L_H given by $g(aH) = (ga)H$ for $g \in G$ and $aH \in L_H$. We call L_H a **left coset G-set**.

5. Let $H < G$. The **centralizer** of H is the set

   $$Z_G(H) := \{g \in G : ghg^{-1} = h \text{ for all } h \in H\},$$

   and the **normalizer** of H is the set

   $$N_G(H) := \{g \in G : gHg^{-1} = H\}.$$

   (a) Show that $N_G(H)$ is the largest subgroup of G in which H is normal.

   (b) Show that $Z_G(H)$ is a normal subgroup of $N_G(H)$.

   (c) Show that the quotient group $N_G(H)/Z_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$.

6. Show that S_3 can never act transitively on a set with 5 elements.

7. Let G be a group which contains an element a whose order is at least 3. Show that $|\mathrm{Aut}(G)| \geq 2$.

8. Let G be a group whose order is a prime power (i.e. a p-**group** for some prime p). Let N be a nontrivial normal subgroup of G. Show that $N \cap Z(G) \neq \{e\}$.

Optional Part

1. Let $\{X_i : i \in I\}$ be a disjoint collection of sets, meaning that $X_i \cap X_j = \emptyset$ for $i \neq j$. Suppose that each X_i is a G-set for the same group G.

   (a) Show that $\bigcup_{i \in I} X_i$ can naturally be viewed as a G-set; we called it the **union** of the G-sets X_i.

   (b) Show that every G-set X is the union of its orbits.

2. Let X and Y be G-sets with the *same* group G. An **isomorphism** between the G-sets X and Y is a bijection $\phi : X \to Y$ which is **equivariant**, i.e. such that $g\phi(x) = \phi(gx)$ for all $x \in X$ and $g \in G$. Two G-sets are **isomorphic** if there exists an equivariant bijection between them.

   Let X be a transitive G-set, and let $x_0 \in X$. Show that X is isomorphic to the G-set L of all left cosets of G_{x_0}. [*Hint:* For $x \in X$, suppose $x = gx_0$, and define $\phi : X \to L$ by $\phi(x) = gG_{x_0}$. Be sure to show that ϕ is well-defined!]

3. Let X_i for $i \in I$ be G-sets for the same group G, and suppose that the sets X_i are not necessarily disjoint. Let $X_i' = \{(x, i) : x \in X_i\}$ for each $i \in I$. Then the sets X_i' are disjoint, and each can still be regarded as a G-set in an obvious way. (The elements of X_i have simply been tagged by i to distinguish them from the elements of X_j for $i \neq j$.) The G-set $\bigcup_{i \in I} X_i'$ is called the **disjoint union** of the G-sets X_i. Show that every G-set is isomorphic to a disjoint union of left coset G-sets. (Therefore, left coset G-sets are *building blocks* of G-sets.)

4. Let G be a group. Show that $G/Z(G)$ is isomorphic to $\mathrm{Inn}(G)$, the set of all inner automorphisms of G. Use this to give another proof of the fact that if $G/Z(G)$ is cyclic, then G is abelian.

5. Let G be a finite group, and let $H \leq G$ be a subgroup of index p, where p is the smallest prime which divides $|G|$.

   (a) Write the action of G on the set G/H of left cosets by left multiplication as a homomorphism $\rho : G \to S_p$, where S_p denotes the p-th symmetric group.

   (b) Show that $\ker \rho \leq H$.

   (c) Further show, by using the hypothesis, that $H = \ker \rho$. Hence, conclude that H is normal in G.

6. Let G be a finite group, and let $H \leq G$ be a subgroup of index n. Prove that H contains a subgroup K which is normal in G and such that $[G : K]$ divides the gcd of $|G|$ and $n!$. [*Hint:* Use the strategy of the preceding exercise.]